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Existence and Uniqueness of Solutions of General Quantum Difference Equations

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Abstract

In this paper, we apply the fixed point method to establish the existence and uniqueness of solutions of the β-initial value problem

$$D_\beta x(t) = g(t, x), \quad x(s_0) = x_0, \ t \in I,$$

associated with the general quantum difference operator D_β, which is defined by $D_\beta f(t) = \big(f(\beta(t)) - f(t)\big)/\big(\beta(t) - t\big)$ for every t with $t \neq \beta(t)$, where β is a strictly increasing continuous function defined on an interval $I \subseteq \mathbb{R}$ that has a unique fixed point $s_0 \in I$ and satisfies the condition $(t - s_0)(\beta(t) - t) \leq 0$ for every $t \in I$. Also, we use the successive approximations method to deduce an expansion form for the β-exponential function.

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1 Introduction

Quantum calculus is receiving an increase of interest due to its applications. For instance, in physics, economics and calculus of variations. It substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions; see, e.g., [1, 2, 5, 11–14]. In [8], we considered a strictly increasing continuous function $\beta : I \to I$ defined on an interval $I \subseteq \mathbb{R}$ and has a unique fixed point $s_0 \in I$, to construct a general quantum difference operator D_β, the β-difference operator, defined by

$$D_\beta f(t) = \begin{cases} \dfrac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ \\ f'(s_0), & t = s_0, \end{cases}$$

where f is an arbitrary function defined on I and is differentiable at $t = s_0$ in the usual sense. The β-difference operator generalizes the well-known quantum difference operators Hahn, q-Jackson and n, q-power difference operator, see [1, 3, 6, 7, 9, 15], and all other such operators. Hence, the calculus associated with D_β is a generalization of the Hahn, q- and the n, q-power calculi and any such calculus, which avoid the repetition of the results in each calculus separately. Also, the forward difference operator on mixed time scale $\Delta_{a,b}$ is a special case of D_β when $\beta(t) = at + b$, $a \geq 1$, $b \geq 0$ and $a + b > 1$, [4].

In [8], we considered our function β when it has only one fixed point $s_0 \in I$ and satisfies the following condition

$$(t - s_0)(\beta(t) - t) \leq 0 \ \text{ for all } t \in I,$$

and gave a rigorous analysis of the calculus based on D_β and its associated integral operator. Some basic properties of such a calculus were stated and proved. For instance, the chain rule, Leibniz' formula, the mean value theorem and the fundamental theorem of β-calculus. Also, in [10] the exponential, trigonometric and hyperbolic functions based on D_β were constructed. Finally, in [9] some basic integral inequalities based on the β-difference operator were investigated, as Hölder's, Minkowski's, Gronwall's and Bernoulli's inequalities.

In proceeding with the calculus based on D_β, this paper is devoted to prove the existence and uniqueness theorems of the solutions of the first order β-initial value problem using the fixed point method. In Section 2, some previous results in the calculus based on D_β, [8], which we need in this paper, are presented. In Section 3, local existence and uniqueness theorems for the solutions of the first order β-initial value problem are established. In Section 4, a global result is given. Also, an expansion form of the β-exponential function is deduced by using the successive approximations method.

In the sequel, $\mathbb{X}$ is a Banach space with norm $\| \cdot \|$, $I \subseteq \mathbb{R}$ is an interval and $s_0 \in I$ is the unique fixed point of β which belongs to I.

2 Preliminaries

In this section, we present some needed results from [8] concerning the calculus associated with D_β.

Definition 2.1. For a function $f : I \to \mathbb{X}$, we define the β-difference operator of f as

$$D_\beta f(t) = \begin{cases} \dfrac{f(\beta(t)) - f(t)}{\beta(t) - t}, & t \neq s_0, \\ \\ f'(s_0), & t = s_0 \end{cases}$$

provided that f' exists at s_0. In this case, we say that $D_\beta f(t)$ is the β-derivative of f at t. We say that f is β-differentiable on I if $f'(s_0)$ exists.

Theorem 2.2. *Assume that $f : I \longrightarrow \mathbb{X}$ and $g : I \longrightarrow \mathbb{R}$ are β-differentiable functions at $t \in I$. Then:*
(i) *The product $fg : I \longrightarrow \mathbb{X}$ is β-differentiable at t and*

$$\begin{aligned} D_\beta(fg)(t) &= (D_\beta f(t))g(t) + f(\beta(t))D_\beta g(t) \\ &= (D_\beta f(t))g(\beta(t)) + f(t)D_\beta g(t). \end{aligned}$$

(ii) *f/g is β-differentiable at t and*

$$D_\beta\big(f/g\big)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))}, \qquad g(t)g(\beta(t)) \neq 0.$$

Lemma 2.3. *The following statements are true.*

(i) The sequence of functions $\{\beta^k\}_{k\in\mathbb{N}_0}$ converges uniformly to the constant function $\hat{\beta} := s_0$ on every compact interval $J \subseteq I$ containing s_0.

(ii) The series $\sum_{k=0}^{\infty} |\beta^k(t) - \beta^{k+1}(t)|$ is uniformly convergent to $|t - s_0|$ on every compact interval $J \subseteq I$ containing s_0.

Lemma 2.4. *Let $f : I \to \mathbb{X}$ be β-differentiable and $D_\beta f(t) = 0$ for all $t \in I$, then $f(t) = f(s_0)$ for all $t \in I$.*

Theorem 2.5. *Assume $f : I \to \mathbb{X}$ is continuous at s_0. Then the function F defined by*

$$F(t) = \sum_{k=0}^{\infty} \Big(\beta^k(t) - \beta^{k+1}(t)\Big) f(\beta^k(t)), \ t \in I \tag{2.1}$$

is a β-antiderivative of f with $F(s_0) = 0$. Conversely, a β-antiderivative F of f vanishing at s_0 is given by (2.1).

Definition 2.6. Let $f : I \to \mathbb{X}$ and $a, b \in I$. We define the β-integral of f from a to b by

$$\int_a^b f(t)d_\beta t = \int_{s_0}^b f(t)d_\beta t - \int_{s_0}^a f(t)d_\beta t, \tag{2.2}$$

where

$$\int_{s_0}^x f(t)d_\beta t = \sum_{k=0}^{\infty} \Big(\beta^k(x) - \beta^{k+1}(x)\Big) f(\beta^k(x)), \quad x \in I \tag{2.3}$$

provided that the series converges at $x = a$ and $x = b$. f is called β-integrable on I if the series converges at a, b for all $a, b \in I$. Clearly, if f is continuous at $s_0 \in I$, then f is β-integrable on I.

Theorem 2.7. *Let $f : I \to \mathbb{X}$ be continuous at s_0. Define the function*

$$F(x) = \int_{s_0}^x f(t)d_\beta t, \quad x \in I. \tag{2.4}$$

Then F is continuous at s_0, $D_\beta F(x)$ exists for all $x \in I$ and $D_\beta F(x) = f(x)$.

Theorem 2.8. *If $f : I \longrightarrow \mathbb{X}$ is β-differentiable on I, then*

$$\int_a^b D_\beta f(t)d_\beta(t) = f(b) - f(a), \quad for\ all\ a, b \in I. \tag{2.5}$$

Definition 2.9. Let $s_0 \in [a, b] \subseteq I$. We define the β-interval by

$$[a, b]_\beta = \{\beta^k(a); k \in \mathbb{N}_0\} \cup \{\beta^k(b); k \in \mathbb{N}_0\} \cup \{s_0\}.$$

For any point $c \in I$, we denote by

$$[c]_\beta = \{\beta^k(c); k \in \mathbb{N}_0\} \cup \{s_0\}.$$

Lemma 2.10. *Let $f : I \to \mathbb{X}$, $g : I \to \mathbb{R}$ be β-integrable functions on I. If*

$$\|f(t)\| \leq g(t) \quad for\ all\ \ t \in [a, b]_\beta,\ a, b \in I, a \leq b,$$

then for $x, y \in [a, b]_\beta,\ \ x < s_0 < y$, *we have*

$$\Big\|\int_{s_0}^y f(t)d_\beta t\Big\| \leq \int_{s_0}^y g(t)d_\beta t, \tag{2.6}$$

$$\Big\|\int_{s_0}^x f(t)d_\beta t\Big\| \leq \int_x^{s_0} g(t)d_\beta t \tag{2.7}$$

and

$$\Big\|\int_x^y f(t)d_\beta t\Big\| \leq \int_x^y g(t)d_\beta t. \tag{2.8}$$

Consequently, if $g(t) \geq 0$ for all $t \in [a, b]_\beta$, then the inequalities $\int_{s_0}^y g(t)d_\beta t \geq 0$ and $\int_x^y g(t)d_\beta t \geq 0$ hold for all $x, y \in [a, b]_\beta, x < s_0 < y,\ a, b \in I, a \leq b$.

The following definition and theorem are results from [10].

Definition 2.11 (β-Exponential Function)**.** Assume that $p : I \to \mathbb{C}$ is a continuous function at s_0. We define the β-exponential function $e_{p,\beta}(t)$ by

$$e_{p,\beta}(t) = \frac{1}{\prod_{k=0}^{\infty}\Big[1 - p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))\Big]}. \tag{2.9}$$

Theorem 2.12. *The β-exponential function $e_{p,\beta}(t)$ is the unique solution of the first order β-difference equation*

$$D_\beta y(t) = p(t)y(t), \qquad y(s_0) = 1. \tag{2.10}$$

Proof. It is obvious that $e_{p,\beta}(s_0) = 1$. We have

$$\begin{aligned}
D_\beta e_{p,\beta}(t) &= \frac{e_{p,\beta}(\beta(t)) - e_{p,\beta}(t)}{\beta(t) - t} \\
&= \frac{1}{\beta(t) - t}\Big[\frac{1}{\prod_{k=0}^{\infty}(1 - p(\beta^{k+1}(t))(\beta^{k+1}(t) - \beta^{k+2}(t)))} \\
&\qquad - \frac{1}{\prod_{k=0}^{\infty}(1 - p(\beta^{k}(t))(\beta^{k}(t) - \beta^{k+1}(t)))}\Big] \\
&= \frac{p(t)}{\prod_{k=o}^{\infty}(1 - p(\beta^{k}(t))(\beta^{k}(t) - \beta^{k+1}(t)))} = p(t)e_{p,\beta}(t).
\end{aligned}$$

To prove the uniqueness of the solution $e_{p,\beta}(t)$, let x be another solution of Equation (2.10). We have

$$D_\beta\Big(\frac{x(t)}{e_{p,\beta}(t)}\Big) = \frac{e_{p,\beta}(t)D_\beta x(t) - x(t)D_\beta e_{p,\beta}(t)}{e_{p,\beta}(t)e_{p,\beta}(\beta(t))} = 0, \;\; t \in I.$$

By Lemma 2.4, $\dfrac{x(t)}{e_{p,\beta}(t)}$ is a constant function and $\dfrac{x(t)}{e_{p,\beta}(t)} = \dfrac{x(s_0)}{e_{p,\beta}(s_0)} = 1$, i.e., $x(t) = e_{p,\beta}(t)$ for all $t \in I$. □

3 Local Existence and Uniqueness

In this section, we use the fixed point method to show the existence and uniqueness of the solutions of the β-initial value problem (β-IVP)

$$D_\beta x(t) = f(t, x), \qquad x(s_0) = x_0, \;\; t \in I. \tag{3.1}$$

Theorem 3.1. *Let $f : \mathbb{U} \subset \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a continuous function at (s_0, x_0), and ϕ be a function defined on an interval $I \subseteq \mathbb{R}$ such that $s_0 \in I$. Then ϕ is a solution of the β-IVP* (3.1) *if, and only if,* *(1) For all* $t \in I$, $(t, \phi(t)) \in \mathbb{U}$.

(2) ϕ *is continuous at* s_0.

(3) For all $t \in I$, $\phi(t) = x_0 + \int_{s_0}^{t} f(\tau, \phi(\tau)) d_\beta \tau$.

Proof. Let ϕ be a solution of the β-IVP (3.1). Then

$$D_\beta \phi(t) = f(t, \phi(t)), \quad \text{for all } t \in I, \tag{3.2}$$

which implies $(t, \phi(t)) \in \mathbb{U}$ for all $t \in I$. Also, since ϕ is β-differentiable on I, then it is continuous at s_0. Finally, integrating both sides of (3.2) from s_0 to t, we get

$$\phi(t) - \phi(s_0) = \int_{s_0}^{t} f(\tau, \phi(\tau)) d_\beta \tau.$$

Then,

$$\phi(t) = x_0 + \int_{s_0}^{t} f(\tau, \phi(\tau)) d_\beta \tau.$$

Conversely, assume the items (1), (2) and (3) are satisfied, then ϕ is ordinary differentiable at s_0. Consequently, it is β-differentiable on I with $D_\beta \phi(t) = f(t, \phi(t))$ and $\phi(s_0) = x_0$. Therefore, ϕ is a solution of the β-IVP (3.1). $\square$

Throughout this paper, $R \subset \mathbb{U}$ is defined by

$$R = \{(t, x) \in I \times \mathbb{X} : |t - s_0| \leq a, \|x - x_0\| \leq b\},$$

where $a, b > 0$. Let $M = \sup_{(t,x) \in R} \|f(t, x)\| < \infty$. In the following theorem, we use the fixed point method to prove the existence and uniqueness of the solution of the β-IVP (3.1).

Theorem 3.2. *Assume that the function* $f : R \to \mathbb{X}$ *is continuous at* $(s_0, x_0) \in R$ *and satisfies the Lipschitz condition (with respect to* x*)*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|, \quad \textit{for all} \ (t, x_1), (t, x_2) \in R. \tag{3.3}$$

Then the β*-IVP has a unique solution on* $[s_0 - \delta, s_0 + \delta]$*, where* L *is a positive constant and* $\delta = \min\left\{a, \frac{b}{Lb + M}, \frac{\rho}{L}\right\}$ *with* $\rho \in (0, 1)$.

Proof. We prove the theorem for $t \in [s_0, s_0 + \delta]$ and the proof for $t \in [s_0 - \delta, s_0]$ is similar. Define the operator T by

$$Tx(t) = x_0 + \int_{s_0}^{t} f(\tau, x(\tau)) d_\beta \tau. \tag{3.4}$$

Let $C_{[s_0,s_0+\delta]}$ be the space of all continuous functions at s_0 and bounded on the interval $[s_0, s_0+\delta]$ with the supremum norm such that for $x \in C_{[s_0,s_0+\delta]}$, $\|x\|_\infty = \sup_{t\in[s_0,s_0+\delta]} \|x(t)\|$. This space is complete. Let $S = \{x \in C_{[s_0,s_0+\delta]} : \|x - x_0\|_\infty \leq b\}$. $S \subset C_{[s_0,s_0+\delta]}$ and is closed, then S is a complete metric space. First, we prove that $T : S \to S$. Let $\phi \in S$,

$$\begin{aligned}
\|T\phi(t) - x_0\| =& \Big\| \int_{s_0}^{t} f(\tau, \phi(\tau)) d_\beta\tau \Big\| \\
=& \Big\| \int_{s_0}^{t} \Big(f(\tau, \phi(\tau)) - f(\tau, x_0) + f(\tau, x_0) \Big) d_\beta\tau \Big\| \\
\leq& \int_{s_0}^{t} \Big\| f(\tau, \phi(\tau)) - f(\tau, x_0) + f(\tau, x_0) \Big\| d_\beta\tau \\
\leq& \int_{s_0}^{t} \Big\| f(\tau, \phi(\tau)) - f(\tau, x_0) \Big\| + \Big\| f(\tau, x_0) \Big\| \; d_\beta\tau \\
\leq& \int_{s_0}^{t} \Big(L\|\phi(\tau) - x_0\| + M \Big) d_\beta\tau \\
\leq& (Lb + M) \int_{s_0}^{t} d_\beta\tau \\
\leq& (Lb + M)(t - s_0) \\
\leq& (Lb + M)\delta.
\end{aligned}$$

In view of $\delta \leq \dfrac{b}{Lb + M}$, we have $\|T\phi(t) - x_0\| \leq b$, i.e., $T\phi \in S$. Second, we show that T is a contraction mapping. Let $\phi_1, \phi_2 \in S$,

$$\begin{aligned}
\|T\phi_1(t) - T\phi_2(t)\| =& \Big\| \int_{s_0}^{t} \Big(f(\tau, \phi_1(\tau)) - f(\tau, \phi_2(\tau)) \Big) d_\beta\tau \Big\| \\
\leq& \int_{s_0}^{t} \Big\| f(\tau, \phi_1(\tau)) - f(\tau, \phi_2(\tau)) \Big\| d_\beta\tau \\
\leq& \int_{s_0}^{t} L \Big\| \phi_1(\tau) - \phi_2(\tau) \Big\| \; d_\beta\tau \\
\leq& L\|\phi_1 - \phi_2\|_\infty \int_{s_0}^{t} d_\beta\tau \\
=& L\|\phi_1 - \phi_2\|_\infty (t - s_0) \\
\leq& L\delta\|\phi_1 - \phi_2\|_\infty \\
\leq& \rho\|\phi_1 - \phi_2\|_\infty.
\end{aligned}$$

Then T is a contraction mapping. By Banach's fixed point theorem, T has a unique fixed point in S and then the β-IVP (3.1) has a unique solution in S. $\square$

4 Global Existence and Uniqueness

In this section, we use the fixed point method to show that, if the function f satisfies the Lipschitz condition on $D = [s_0, s_0 + a] \times \mathbb{X}$ rather than on R, then the β-IVP (3.1) has a unique solution on the entire interval $[s_0, s_0 + a]$.

Theorem 4.1. *Let f be continuous at (s_0, x_0) and satisfies the Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\| \quad \textit{for all } (t, x_1), (t, x_2) \in D,$$

with $L < \dfrac{1}{a}$, then the β-IVP (3.1) *has a unique solution on the entire interval* $[s_0, s_0+a]$.

Proof. Let the operator T be as defined in (3.4), $t \in [s_0, s_0+a]$ and suppose $C_{[s_0,s_0+a]}$ is the complete metric space of all continuous functions at s_0 and bounded on the interval $[s_0, s_0 + a]$ with the supremum norm. Let $\phi(t) \in C_{[s_0,s_0+a]}$. Then it is clear that $T\phi(t) \in C_{[s_0,s_0+a]}$. To show that T is contraction, assume that $\phi_1, \phi_2 \in C_{[s_0,s_0+a]}$. Then

$$\begin{aligned}
||T\phi_1(t) - T\phi_2(t)|| &= \Big\| \int_{s_0}^{t} \Big(f(\tau, \phi_1(\tau)) - f(\tau, \phi_2(\tau))\Big) d_\beta\tau \Big\| \\
&\leq \int_{s_0}^{t} \Big\|f(\tau, \phi_1(\tau)) - f(\tau, \phi_2(\tau))\Big\| d_\beta\tau \\
&\leq \int_{s_0}^{t} L\Big\|\phi_1(\tau) - \phi_2(\tau)\Big\| \, d_\beta\tau \\
&= L||\phi_1 - \phi_2||_\infty \int_{s_0}^{t} d_\beta\tau \\
&= L||\phi_1 - \phi_2||_\infty (t - s_0) \\
&\leq La||\phi_1 - \phi_2||_\infty.
\end{aligned}$$

where $La < 1$. Then , T has a unique fixed point in $C_{[s_0,s_0+a]}$ and then the β-IVP (3.1) has a unique solution in $C_{[s_0,s_0+a]}$. □

One can see that the successive approximation method, [7, 16], can be used to prove the existence and uniqueness of the solution of the β-IVP (3.1).

Theorem 4.2. *Assume that $f : R \to \mathbb{X}$ is continuous at (s_0, x_0) and satisfies the Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|, \quad \textit{for all } \ (t, x_1), (t, x_2) \in R$$

where L is a positive constant. Then the sequence defined by

$$\phi_{k+1}(t) = x_0 + \int_{s_0}^{t} f(\tau, \phi_k(\tau)) d_\beta\tau, \quad \phi_0(t) = x_0, \quad |t - s_0| \leq \delta, \ \ k \geq 0 \tag{4.1}$$

converges uniformly on the interval $|t - s_0| \leq \delta$ to, a function ϕ, the unique solution of the β-IVP (3.1)*, where $\delta = \min\left\{a, \dfrac{b}{Lb + M}, \dfrac{\rho}{L}\right\}$ with $\rho \in (0, 1)$.*

The following theorem shows that if the function f satisfies the Lipschitz condition on $D = [s_0 - a, s_0 + a] \times \mathbb{X}$ rather than on R, then the solutions will exist on the entire interval $[s_0 - a, s_0 + a]$.

Theorem 4.3. *Let f be continuous at (s_0, x_0) and satisfies the Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\| \quad \textit{for all } (t, x_1), (t, x_2) \in D,$$

where $L < \dfrac{1}{a}$, then the successive approximations ϕ_k that are given in (4.1) *converge uniformly on $[s_0 - a, s_0 + a]$ to the unique solution of the β-IVP* (3.1).

In the following, we deduce an expansion form of the β-exponential function (2.9) using the method of successive approximations.

Theorem 4.4. *Let $z \in \mathbb{C}$ be a constant. Then the function ϕ defined by*

$$\phi(t) = \sum_{k=0}^{\infty} z^k \alpha_k(t), \tag{4.2}$$

is the unique solution of the β-IVP

$$D_\beta x(t) = zx(t), \;\; x(s_0) = 1, \tag{4.3}$$

where

$$\alpha_k(t) = \begin{cases} \displaystyle\sum_{i_1, i_2, i_3, \ldots, i_{k-1}=0}^{\infty} \Big(\prod_{l=1}^{k-1} (\beta, \beta)_{\sum_{j=1}^{l} i_j}\Big)\Big(\beta^{\sum_{j=1}^{k-1} i_j}(t) - s_0\Big), & \textit{if} \;\; k \geq 2, \\ t - s_0, & \textit{if} \;\; k = 1, \\ 1, & \textit{if} \;\; k = 0, \end{cases}$$

with $(\beta, \beta)_i = \beta^i(t) - \beta^{i+1}(t)$.

Proof. We prove by the method of successive approximations. Choose $\phi_0(t) = 1$ as the initial solution. Then using relation (4.1) to get

$$\begin{aligned} \phi_1(t) =& 1 + \int_{s_0}^{t} f(\tau, \phi_0(\tau)) d_\beta \tau \\ &= 1 + \int_{s_0}^{t} z\phi_0(\tau) d_\beta \tau \\ &= 1 + z(t - s_0). \end{aligned}$$

Again by relation (4.1), the function ϕ_2 is given by

$$\begin{aligned}\phi_2(t) =&1+\int_{s_0}^{t} f(\tau,\phi_1(\tau))d_\beta\tau\\ &=1+\int_{s_0}^{t} z\phi_1(\tau)d_\beta\tau\\ &=1+\int_{s_0}^{t}\Big(z+z^2(\tau-s_0)\Big)d_\beta\tau\\ &=1+z(t-s_0)+z^2\sum_{i_1=0}^{\infty}\Big(\beta^{i_1}(t)-\beta^{i_1+1}(t)\Big)\Big(\beta^{i_1}(t)-s_0\Big)\\ &=1+z(t-s_0)+z^2\sum_{i_1=0}^{\infty}\Big(\beta,\beta\Big)_{i_1}\Big(\beta^{i_1}(t)-s_0\Big).\end{aligned}$$

Also, $\phi_3(t)$ is given by

$$\begin{aligned}\phi_3(t) &= 1+\int_{s_0}^{t} f(\tau,\phi_2(\tau))d_\beta\tau\\ &=1+\int_{s_0}^{t} z\phi_2(\tau)d_\beta\tau\\ &=1+z(t-s_0)+z^2\sum_{i_1=0}^{\infty}\Big(\beta,\beta\Big)_{i_1}\Big(\beta^{i_1}(t)-s_0\Big)\\ &\qquad+z^3\sum_{i_2=0}^{\infty}\Big(\beta,\beta\Big)_{i_2}\sum_{i_1=0}^{\infty}\Big(\beta,\beta\Big)_{i_1+i_2}\Big(\beta^{i_1+i_2}(t)-s_0\Big)\\ &=1+z(t-s_0)+z^2\sum_{i_1=0}^{\infty}\Big(\beta,\beta\Big)_{i_1}\Big(\beta^{i_1}(t)-s_0\Big)\\ &\qquad+z^3\sum_{i_2,i_1=0}^{\infty}\Big(\beta,\beta\Big)_{i_2}\Big(\beta,\beta\Big)_{i_1+i_2}\Big(\beta^{i_1+i_2}(t)-s_0\Big).\end{aligned}$$

By induction on n, we can show that

$$\phi_n(t)=1+z(t-s_0)+\sum_{k=2}^{n} z^k\sum_{i_1,i_2,i_3,\ldots,i_{k-1}=0}^{\infty}\Big(\prod_{l=1}^{k-1}(\beta,\beta)_{\sum_{j=1}^{l} i_j}\Big)\Big(\beta^{\sum_{j=1}^{k-1} i_j}(t)-s_0\Big).$$

So, by Theorem 4.3 $\phi_n\to\phi=\sum_{k=0}^{\infty} z^k\alpha_k(t)$. □

Example 4.5. Let $\beta(t) = qt$, $q \in (0,1)$. Then $s_0 = 0$ and $\beta^i(t) = q^i t$, $i = 0,1,2,\cdots$. In this case $(\beta,\beta)_i = q^i t(1-q)$. Therefore,

$$\alpha_k(t) = \begin{cases} \displaystyle\sum_{i_1,i_2,i_3,\ldots,i_{k-1}=0}^{\infty} \Big(\prod_{l=1}^{k-1} t(1-q)q^{\sum_{j=1}^{l} i_j}\Big)\Big(tq^{\sum_{j=1}^{k-1} i_j}\Big), & \text{if} \quad k \geq 2, \\ t - s_0, & \text{if} \quad k = 1, \\ 1, & \text{if} \quad k = 0. \end{cases}$$

We show by induction that

$$\alpha_k(t) = \frac{(t(1-q))^k}{(q;q)_k}, \quad k = 0,1,2,\cdots, \tag{4.4}$$

where

$$(q;q)_k = \begin{cases} \displaystyle\prod_{m=1}^{k}(1-q^m), & \text{if} \quad k \in \mathbb{N}, \\ 1, & \text{if} \quad k = 0. \end{cases}$$

For $k = 0$, $\alpha_0(t) = 1$. For $k = 1$, $\alpha_1(t) = \dfrac{t(1-q)}{1-q} = t$. For $k = 2$, $\alpha_2(t) = t^2(1-q)\displaystyle\sum_{i_1=0}^{\infty} q^{2i_1} = \frac{t^2(1-q)}{1-q^2}$. Suppose that (4.4) is true for $k = m \geq 2$. Then for $k = m+1$, we have

$$\begin{aligned} \alpha_{m+1}(t) &= \sum_{i_1,i_2,i_3,\ldots,i_m=0}^{\infty} \Big(\prod_{l=1}^{m} t(1-q)q^{\sum_{j=1}^{l} i_j}\Big)\Big(tq^{\sum_{j=1}^{m} i_j}\Big) \\ &= t^{m+1}(1-q)^m \sum_{i_1=0}^{\infty} q^{(m+1)i_1} \sum_{i_2=0}^{\infty} q^{mi_2} \cdots \sum_{i_m=0}^{\infty} q^{2i_m} \\ &= \frac{t^{m+1}(1-q)^m}{(1-q^2)(1-q^3)\cdots(1-q^m)(1-q^{m+1})} \\ &= \frac{(t(1-q))^{m+1}}{(q,q)_{m+1}}. \end{aligned}$$

Consequently,

$$\phi(t) = \sum_{k=0}^{\infty} \frac{(zt(1-q))^k}{(q;q)_k},$$

which is the expansion of q-exponential function, the unique solution of (4.3). See [1,11].

Example 4.6. Let $\beta(t) = qt + \omega$, $q \in (0,1)$, $\omega > 0$. Then $s_0 = \dfrac{\omega}{1-q}$ and $\beta^i(t) = q^i t + \omega\dfrac{1-q^i}{1-q}$. So, $(\beta,\beta)_i = q^i\Big(t(1-q) - \omega\Big)$ and $(\beta^i(t) - s_0) = \dfrac{q^i(t(1-q)-\omega)}{1-q}$. Therefore,

$$\alpha_k(t) = \begin{cases} \displaystyle\sum_{i_1,i_2,i_3,\ldots,i_{k-1}=0}^{\infty} \Big(\prod_{l=1}^{k-1}(t(1-q)-\omega)q^{\sum_{j=1}^{l} i_j}\Big)\Big(\frac{(t(1-q)-\omega)}{1-q}q^{\sum_{j=1}^{k-1} i_j}\Big), \\ \qquad\qquad \text{if } k \geq 2, \\ t - s_0, \quad \text{if } k = 1, \\ 1, \quad\quad\;\; \text{if } k = 0. \end{cases}$$

We prove by induction that

$$\alpha_k(t) = \frac{(t(1-q)-\omega)^k}{(q;q)_k}, \quad k = 0,1,2,\cdots. \tag{4.5}$$

For $k = 0$, $\alpha_0(t) = 1$. For $k = 1$, $\alpha_1(t) = \dfrac{t(1-q)-\omega}{1-q} = t - s_0$. For $k = 2$, $\alpha_2(t) = \dfrac{(t(1-q)-\omega)^2}{(1-q)(1-q^2)}$. Suppose that (4.5) is true for $k = m$. Then for $k = m+1$, we have

$$\begin{aligned} \alpha_{m+1}(t) = &\sum_{i_1,i_2,\cdots,i_m=0}^{\infty} \Big(q^{i_1}\Big(t(1-q)-\omega\Big)\Big)\Big(q^{i_1+i_2}\Big(t(1-q)-\omega\Big)\Big)\cdots \\ &\Big(q^{i_1+i_2+\cdots+i_m}\Big(t(1-q)-\omega\Big)\Big)\Big(\frac{q^{i_1+i_2+\cdots+i_m}(t(1-q)-\omega)}{1-q}\Big) \\ = &\frac{(t(1-q)-\omega)^{m+1}}{1-q}\sum_{i_1=0}^{\infty} q^{(m+1)i_1}\sum_{i_2=0}^{\infty} q^{mi_2}\cdots\sum_{i_m=0}^{\infty} q^{2i_m} \\ = &\frac{(t(1-q)-\omega)^{m+1}}{(1-q)(1-q^2)\cdots(1-q^m)(1-q^{m+1})} \\ = &\frac{(t(1-q)-\omega)^{m+1}}{(q,q)_{m+1}}. \end{aligned}$$

Hence,

$$\phi(t) = \sum_{k=0}^{\infty}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k},$$

which is the Hahn-exponential function, the unique solution of (4.3). See [1].

We combine Theorem 2.12 and Theorem 4.4 to obtain the following result.

Proposition 4.7. *Let $z \in \mathbb{C}$. The β-exponential function $e_{z,\beta}(t)$ has the expansion*

$$e_{z,\beta}(t) = \sum_{k=0}^{\infty} z^k \alpha_k(t), \tag{4.6}$$

$$\alpha_k(t) = \begin{cases} \displaystyle\sum_{i_1,i_2,i_3,\ldots,i_{k-1}=0}^{\infty} \Big(\prod_{l=1}^{k-1} (\beta,\beta)_{\sum_{j=1}^{l} i_j}\Big)\Big(\beta^{\sum_{j=1}^{k-1} i_j}(t) - s_0\Big), & \textit{if} \quad k \geq 2, \\ t - s_0, & \textit{if} \quad k = 1, \\ 1, & \textit{if} \quad k = 0. \end{cases}$$

5 Conclusion

This paper was devoted to use the fixed point method for proving the existence and uniqueness of solutions of the β-initial value problem associated with the β-difference operator which is defined by $D_\beta f(t) = \big(f(\beta(t)) - f(t)\big)/\big(\beta(t) - t\big)$, for every t with $t \neq \beta(t)$ where f is an arbitrary function defined on $I \subset \mathbb{R}$ and β is a strictly increasing continuous function defined on I and satisfies the condition $(t - s_0)(\beta(t) - t) \leq 0$ for every $t \in I$. Also, an expansion form of the β-exponential function was deduced by the successive approximations method.

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