Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 11, Number 1, pp. 33–43 (2016)
http://campus.mst.edu/adsa

Lyapunov-Type Inequalities for some Sequential Fractional Boundary Value Problems

Rui A. C. Ferreira
Grupo Física-Matemática
Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal.
raferreira@fc.ul.pt

Abstract

In this work, we consider a sequential fractional differential equation subject to Dirichlet-type boundary conditions and we perform an analysis aiming to derive a Lyapunov-type inequality. We cover the cases of the researcher's most used fractional differential operators, namely, the Riemann–Liouville and the Caputo ones. As an application of our results, we present criteria for the nonexistence of real zeros to a generalized sine function.

AMS Subject Classifications: 34A08, 34A40, 26D10, 34C10.
Keywords: Sequential fractional derivative, Lyapunov inequality, special functions.

1 Introduction

The famous Lyapunov inequality, named after the Russian mathematician A. M. Lyapunov used it to study the stability of solutions of second order differential equations, can be stated as follows:

Theorem 1.1 (See [9]). *If the boundary value problem*

$$\begin{aligned} y''(t) + q(t)y(t) = 0, &\quad a < t < b, \\ y(a) = 0 = y(b), & \end{aligned}$$

has a nontrivial continuous solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.1}$$

Received June 29, 2016; Accepted August 29, 2016
Communicated by Martin Bohner

Typical applications of this result include bounds for eigenvalues, stability criteria for periodic differential equations, and estimates for intervals of disconjugacy. Thus, it is not surprising that scholars have written surveys and books in which Lyapunov-type inequalities play a major role [2, 13].

The author succeeded to generalize Theorem 1.1 for boundary value problems in which the classical derivative y'' is replaced by a fractional derivative ${}_a\mathcal{D}^\alpha$ (cf. the definitions in Section 2), namely, the Riemann–Liouville fractional derivative or the Caputo fractional derivative (cf. [4] and [5], respectively).

In this work we aim to obtain a Lyapunov-type inequality for a different problem from those cited above. Indeed, we will consider the following *sequential fractional* boundary value problem,

$$({}_a\mathcal{D}^\alpha {}_a\mathcal{D}^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \tag{1.2}$$

$$y(a) = 0 = y(b), \tag{1.3}$$

where α, β are some real numbers to be defined later and ${}_a\mathcal{D}^\alpha$ stands for the Riemann–Liouville or the Caputo fractional derivative. Some applications of linear ordinary sequential fractional differential equations may be found in [8, Section 7.6] while initial developments on these (sequential) operators may be consulted in the book [10].

At a first glance one might think that because the fractional boundary value problem (1.2)–(1.3) has only a minor change in its formulation, when compared to the ones studied in [4, 5], then the analysis will be somehow the same. However, our experience in dealing with this subject taught us that changing (only) the fractional operator may lead to a more complex analysis of the problem (cf. the differences between the Riemann–Liouville and the Caputo cases in [4] and [5], respectively). Let us now make a brief description of why this happens: To obtain a Lyapunov-type inequality for a fractional boundary value problem we use a method that goes back at least to Nehari [11]. The method consists in transforming the BVP into an equivalent integral form and then find the maximum of the modulus of its Green's function. The advantage is that then you will need not have to use properties of the classical calculus that do not necessarily hold within the fractional calculus setting (cf. Borg's proof of the Lyapunov inequality in [2, Section 1]); we will only be dealing with an integral equation and, in some sense, forget about fractional calculus. To make things more precise, suppose that a given boundary value problem is equivalent to the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where G stands for the Green's function, as known usually in the literature. Then, assuming y is nontrivial (which implies that q is not the zero function and G is not constant in view that in our study $G(a,s) = 0$, $s \in [a,b]$), we get

$$\|y\| < \max_{(t,s)\in[a,b]\times[a,b]} |G(t,s)| \int_a^b |q(s)|ds\|y\|,$$

or

$$\frac{1}{\max_{(t,s)\in[a,b]\times[a,b]}|G(t,s)|} < \int_a^b |q(s)|ds, \tag{1.4}$$

being (1.4) the desired Lyapunov inequality. The drawback is that the Green's function might be very complex to analyze[1], e.g. so far we are aware of results for fractional differential equations with order between one and two and one of order between three and four [4–7, 12, 15], in contrast to the classical case (cf. [13, Section 2.1.1.3]).

The plan of this manuscript is as follows: in Section 2 we provide to the reader a brief introduction to some of the fractional calculus concepts and results. In Section 3 we prove our results and finally, in Section 4, we present an example of application of one of our Lyapunov-type inequalities.

2 Fractional Calculus

We will introduce the concepts as well as some results used throughout this work.

Definition 2.1. Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann–Liouville fractional integral of order α is defined by $(I_q^0 f)(x) = f(x)$ and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a,b].$$

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined by

$$({}_aD^0 f)(t) = f(t) \text{ and } ({}_aD^\alpha f)(t) = (D^m {}_aI^{m-\alpha} f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater or equal than α.

Definition 2.3. The Caputo fractional derivative of order $\alpha \geq 0$ is defined by

$$({}_a^CD^0 f)(t) = f(t) \text{ and } ({}_a^CD^\alpha f)(t) = ({}_aI^{m-\alpha} D^m f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater or equal than α.

The following sequence of results may be found in the book by Kilbas *et al.* [8].

Proposition 2.4. *Let f be a continuous function on some interval I and $\alpha, \beta > 0$. Then*

$$({}_aI^\alpha {}_aI^\beta f)(t) = ({}_aI^{\alpha+\beta} f)(t) = ({}_aI^\beta {}_aI^\alpha f)(t) \text{ on } I.$$

Proposition 2.5. *Let f be a continuous function on some interval I and $\alpha > 0$. Then*

$$({}_a\mathcal{D}^\alpha {}_aI^\alpha f)(t) = f(t) \text{ on } I,$$

with ${}_a\mathcal{D}$ being the Riemann–Liouville or the Caputo fractional operator.

[1] We will comment about the difficulties we had to obtain the Lyapunov inequality for the Caputo case when compared to the one studied in [5].

Proposition 2.6. *The general solution y of the following fractional differential equation*

$$({}_aD^\alpha y)(t) = f(t),\ t > a,\ 0 < \alpha \le 1,$$

is $y(t) = c(t-a)^{\alpha-1} + ({}_aI^\alpha f)(t),\ c \in \mathbb{R}$.

Proposition 2.7. *The general solution y of the following fractional differential equation*

$$({}_a^CD^\alpha y)(t) = f(t),\ t \ge a,\ 0 < \alpha \le 1,$$

is $y(t) = c + ({}_aI^\alpha f)(t),\ c \in \mathbb{R}$.

3 Main Results

In this section we present and prove our main accomplishments. For the reader's benefit we provide the results in two different subsections, the first one containing the results using the Riemann–Liouville fractional differential operator and the second one using the Caputo fractional differential operator.

3.1 Riemmann–Liouville's Case

Let $0 < \alpha, \beta \le 1$ and fix $1 < \gamma := \alpha + \beta \le 2$. Moreover, assume that q is a real valued continuous function on $[a, b]$.

Deriving a Lyapunov-type inequality for the boundary value problem

$$({}_aD^\alpha {}_aD^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \tag{3.1}$$

$$y(a) = 0 = y(b), \tag{3.2}$$

is actually easy because we can use [4, Theorem 2.1] in this case. Indeed, assuming that (3.1)–(3.2) has a nontrivial solution $y \in C[a, b]$, then it is of the form

$$y(t) = c\frac{\Gamma(\alpha)}{\Gamma(\gamma)}(t-a)^{\gamma-1} - ({}_aI^\gamma qy)(t),$$

by Proposition 2.6 and the fact that ${}_aI^\beta(t-a)^{\alpha-1} = \dfrac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}$ ($c \in \mathbb{R}$ is determined by the condition $y(b) = 0$). It is clear that y' is integrable on $[a, b]$. Then (see [14, Section 2.3.6–2.3.7]),

$$({}_aD^\alpha {}_aD^\beta y)(t) = ({}_aD^\gamma y)(t).$$

The following result is therefore an immediate consequence of [4, Theorem 2.1].

Theorem 3.1. *If the fractional boundary value problem* (3.1)–(3.2) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds > \Gamma(\gamma) \left(\frac{4}{b-a} \right)^{\gamma-1}. \tag{3.3}$$

Remark 3.2. Note that if $\alpha = \beta = 1$, i.e., $\gamma = 2$, then we get the classical Lyapunov inequality (1.1).

3.2 Caputo's Case

We consider now the BVP (1.2)–(1.3) with ${}_a\mathcal{D}^\delta ={}_a^C D^\delta$, for $\delta \in \{\alpha, \beta\}$.

Remark 3.3. Contrarily to the Riemann–Liouville case, for a continuous function y the equality $({}_a^C D^\alpha {}_a^C D^\beta y)(t) = ({}_a^C D^{\alpha+\beta} y)(t)$ does not hold in general (see [3, Remark 3.4. (b)]).

Lemma 3.4. *Let* $0 < \alpha, \beta \leq 1$ *be such that* $1 < \alpha + \beta \leq 2$ *and* $q \in C[a, b]$ *for some* $a < b$. *Then,* $y \in C[a, b]$ *is a solution of the fractional boundary value problem*

$$({}_a^C D^\alpha {}_a^C D^\beta y)(t) + q(t)y(t) = 0, \quad a < t < b, \tag{3.4}$$

$$y(a) = 0 = y(b), \tag{3.5}$$

if, and only if, y *satisfies the integral equation*

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds,$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha + \beta)} \begin{cases} \dfrac{(b-s)^{\alpha+\beta-1}(t-a)^\beta}{(b-a)^\beta} - (t-s)^{\alpha+\beta-1}, & a \leq s \leq t \leq b, \\ \dfrac{(b-s)^{\alpha+\beta-1}(t-a)^\beta}{(b-a)^\beta}, & a \leq t \leq s \leq b. \end{cases}$$

Proof. The proof is just a repeated application of Proposition 2.7 and the concomitant use of the boundary conditions. We leave the details to the reader. □

The rest of this section is devoted essentially to determine the maximum of $|G(t, s)|$ for $(t, s) \in [a, b] \times [a, b]$. We will not write it as a "proof" since we will comment on the difficulties and challenges when tackling this problem, which constitute the main differences to the ones studied before by the author [4, 5].

Let us start by defining a function:

$$g_1(t, s) = \frac{(b-s)^{\alpha+\beta-1}(t-a)^\beta}{(b-a)^\beta}, \quad a \leq t \leq s \leq b.$$

g_1 is obviously nonnegative. Moreover,

$$g_1(t,s) \leq \frac{(b-s)^{\alpha+\beta-1}(s-a)^\beta}{(b-a)^\beta} := f(s), \quad s \in [a,b].$$

So, we are left to find $\max\limits_{s\in[a,b]} f(s)$. We have,

$$\begin{aligned}
f'(s) &= \frac{1}{(b-a)^\beta}[-(\alpha+\beta-1)(b-s)^{\alpha+\beta-2}(s-a)^\beta + (b-s)^{\alpha+\beta-1}\beta(s-a)^{\beta-1}] \\
&= \frac{(b-s)^{\alpha+\beta-2}(s-a)^{\beta-1}}{(b-a)^\beta}[-(\alpha+\beta-1)(s-a)+\beta(b-s)],
\end{aligned}$$

from which follows that $f'(s) = 0$ if, and only if, $s = \dfrac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1}$ and $f'(s) > 0$ for $s < \dfrac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1}$ and $f'(s) < 0$ for $s > \dfrac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1}$. We conclude that,

$$\begin{aligned}
\max_{(t,s)\in[a,b]\times[a,b]} |g_1(t,s)| &= f\left(\frac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1}\right) \\
&= \frac{\left(b - \frac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1}\right)^{\alpha+\beta-1}\left(\frac{(\alpha+\beta-1)a+\beta b}{\alpha+2\beta-1} - a\right)^\beta}{(b-a)^\beta} \\
&= \frac{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}{(\alpha+2\beta-1)^{\alpha+2\beta-1}}(b-a)^{\alpha+\beta-1}. \qquad (3.6)
\end{aligned}$$

Now we define a function g_2 by:

$$g_2(t,s) = \frac{(b-s)^{\alpha+\beta-1}(t-a)^\beta}{(b-a)^\beta} - (t-s)^{\alpha+\beta-1}, \quad a \leq s \leq t \leq b.$$

This function resembles the corresponding one in [5]. The substantial difference is that the factor $(t-a)^\delta$ in [5] is of order $\delta = 1$ while here is of *order* $\delta = \beta$. This fact allowed us to determine immediately in [5] the zero of the derivative of g_2 with respect to t for an arbitrary but fixed s (cf. [5, Lemma 2] for the details). However, in this work we cannot proceed as in [5] in view that

$$\partial_t g_2(t,s) = \frac{(b-s)^{\alpha+\beta-1}\beta(t-a)^{\beta-1}}{(b-a)^\beta} - (\alpha+\beta-1)(t-s)^{\alpha+\beta-2}. \qquad (3.7)$$

Fortunately we can use the Fritz John theorem and take advantage of some kind of symmetry of the partial derivatives of our problem in order to find the candidates to maxima of the function $g_2(t,s)$ with $a < s < t < b$. We have

$$\partial_t g_2(t,s) = \frac{(b-s)^{\alpha+\beta-1}\beta(t-a)^{\beta-1}}{(b-a)^{\beta}} - (\alpha+\beta-1)(t-s)^{\alpha+\beta-2} = 0, \tag{3.8}$$

$$\partial_s g_2(t,s) = \frac{-(\alpha+\beta-1)(b-s)^{\alpha+\beta-2}(t-a)^{\beta}}{(b-a)^{\beta}} + (\alpha+\beta-1)(t-s)^{\alpha+\beta-2} = 0, \tag{3.9}$$

from which follows that

$$\frac{(b-s)^{\alpha+\beta-2}(t-a)^{\beta-1}}{(b-a)^{\beta}}[\beta(b-s) - (\alpha+\beta-1)(t-a)] = 0,$$

or

$$t = a + \frac{\beta(b-s)}{\alpha+\beta-1}, \tag{3.10}$$

provided $s < t < b$. Note that it is very hard (if not impossible) to find the solutions in s from inserting (3.10) in (3.8) or (3.9). Therefore, our approach consists in inserting (3.10) in g_2 and then perform an analysis of the resulting function of only the variable s. Before we start the mentioned analysis we need to find out which interval s belongs. On one hand we have,

$$\begin{aligned} s < t \Leftrightarrow s &< a + \frac{\beta(b-s)}{\alpha+\beta-1} \\ \Leftrightarrow s &< \frac{a(\alpha+\beta-1)+\beta b}{\alpha+2\beta-1}. \end{aligned}$$

On the other hand we get,

$$t < b \Leftrightarrow a + \frac{\beta(b-s)}{\alpha+\beta-1} < b \tag{3.11}$$

$$\Leftrightarrow s > \frac{a(\alpha+\beta-1)+b(1-\alpha)}{\beta}. \tag{3.12}$$

Let us finally consider the following function:

$$\begin{aligned} F(s) :&= \frac{(b-s)^{\alpha+\beta-1}\left(a+\frac{\beta(b-s)}{\alpha+\beta-1}-a\right)^{\beta}}{(b-a)^{\beta}} - \left(a+\frac{\beta(b-s)}{\alpha+\beta-1}-s\right)^{\alpha+\beta-1}, \\ &= \frac{(b-s)^{\alpha+2\beta-1}\beta^{\beta}}{(\alpha+\beta-1)^{\beta}(b-a)^{\beta}} - \left(a+\frac{\beta(b-s)}{\alpha+\beta-1}-s\right)^{\alpha+\beta-1}, \end{aligned}$$

with $\dfrac{a(\alpha+\beta-1)+b(1-\alpha)}{\beta} < s < \dfrac{a(\alpha+\beta-1)+\beta b}{\alpha+2\beta-1}$. Fortunately we needed not to calculate zeros of the derivative for this function as it happens that it is strictly increasing. To prove this last statement we start differentiating F:

$$F'(s) = -\frac{(\alpha+2\beta-1)(b-s)^{\alpha+2\beta-2}\beta^\beta}{(\alpha+\beta-1)^\beta(b-a)^\beta} + (\alpha+2\beta-1)\left(a+\frac{\beta(b-s)}{\alpha+\beta-1}-s\right)^{\alpha+\beta-2}.$$

By (3.11) and the fact that $\alpha+\beta-2<0$ and $\alpha+2\beta-1>0$, we get that

$$\begin{aligned}F'(s) &> -\frac{(\alpha+2\beta-1)(b-s)^{\alpha+2\beta-2}\beta^\beta}{(\alpha+\beta-1)^\beta(b-a)^\beta} + (\alpha+2\beta-1)\,(b-s)^{\alpha+\beta-2}\\ &= (\alpha+2\beta-1)(b-s)^{\alpha+\beta-2}\left(1-\frac{(b-s)^\beta\beta^\beta}{(\alpha+\beta-1)^\beta(b-a)^\beta}\right)\\ &> 0,\end{aligned}$$

where the last inequality follows from (3.12). Therefore,

$$\begin{aligned}\max_{s\in\left[\frac{a(\alpha+\beta-1)+b(1-\alpha)}{\beta},\frac{a(\alpha+\beta-1)+\beta b}{\alpha+2\beta-1}\right]}|F(s)| &= F\left(\frac{a(\alpha+\beta-1)+\beta b}{\alpha+2\beta-1}\right),\\ &= \frac{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}{(\alpha+2\beta-1)^{\alpha+2\beta-1}}(b-a)^{\alpha+\beta-1},\end{aligned}$$

in view that $F\left(\frac{a(\alpha+\beta-1)+b(1-\alpha)}{\beta}\right)=0$.

The following result is therefore valid.

Proposition 3.5. *The function G defined in Lemma 3.4 satisfies the following property:*

$$|G(t,s)| \le \frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\frac{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}{(\alpha+2\beta-1)^{\alpha+2\beta-1}},\ (t,s)\in[a,b]\times[a,b],$$

with equality if and only if

$$t = s = \frac{a(\alpha+\beta-1)+\beta b}{\alpha+2\beta-1}.$$

Theorem 3.6. *If the fractional boundary value problem* (3.4)–(3.5) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}}\frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}.$$

Remark 3.7. Observe that when $\alpha=\beta=1$, then Theorem 3.6 reduces to Theorem 1.1. Again this shows that Theorem 3.6 is a generalization of Theorem 1.1.

4 An Example of Application

We will end this work presenting an application of Theorem 3.1. Consider the following sequential fractional differential equation

$$({}_0D^{\alpha}{}_0D^{\alpha}y)(t) + \lambda^2 y(t) = 0,\ \lambda \in \mathbb{R},\ t \in (0,1),\ \frac{1}{2} < \alpha \leq 1. \tag{4.1}$$

The fundamental set of solutions to (4.1) is (cf. [1, Example 1])

$$\{\cos_\alpha(\lambda t), \sin_\alpha(\lambda t)\},$$

where[2]

$$\cos_\alpha(\lambda t) = \sum_{j=0}^{\infty} (-1)^j \lambda^{2j} \frac{t^{(2j+1)\alpha-1}}{\Gamma((2j+1)\alpha)},$$

and

$$\sin_\alpha(\lambda t) = \sum_{j=0}^{\infty} (-1)^j \lambda^{2j+1} \frac{t^{(j+1)2\alpha-1}}{\Gamma((j+1)2\alpha)}.$$

Therefore the general solution of (4.1) is

$$y(t) = c\cos_\alpha(\lambda t) + d\sin_\alpha(\lambda t). \tag{4.2}$$

Now, the nontrivial solutions of (4.2) for which the boundary conditions $y(0) = 0 = y(1)$ hold, satisfy

$$\sin_\alpha(\lambda) = 0,$$

where λ is a real number different from zero (eigenvalue). By Theorem 3.1, the following inequality then holds

$$\lambda^2 > \Gamma(2\alpha)4^{2\alpha-1},$$

or in other words:

Theorem 4.1. *Let* $\frac{1}{2} < \alpha \leq 1$. *If*

$$|t| \leq \sqrt{\Gamma(2\alpha)4^{2\alpha-1}}, \quad t \neq 0,$$

then $\sin_\alpha(t)$ *has not real zeros.*

Remark 4.2. Analogous results to the ones in this section can be obtained if we use the Caputo fractional operator and the trigonometric functions as defined in [1, Example 2].

[2] The definitions of $\cos_\alpha$ and $\sin_\alpha$ in [1] should be as we are writing them here.

Acknowledgements

The author was supported by the “Fundação para a Ciência e a Tecnologia (FCT)” through the program “Investigador FCT” with reference IF/01345/2014.

References

[1] B. Bonilla, M. Rivero and J. J. Trujillo, Linear differential equations of fractional order, in *Advances in fractional calculus*, 77–91, Springer, Dordrecht.

[2] R. C. Brown and D. B. Hinton, Lyapunov inequalities and their applications, in *Survey on classical inequalities*, 1–25, Math. Appl., 517 Kluwer Acad. Publ., Dordrecht.

[3] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, 2004, Springer, Berlin, 2010.

[4] R. A. C. Ferreira, A Lyapunov-type inequality for a fractional boundary value problem, Fract. Calc. Appl. Anal. 16 (2013), no. 4, 978–984.

[5] R. A. C. Ferreira, On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function, J. Math. Anal. Appl. 412 (2014), no. 2, 1058–1063.

[6] M. Jleli and B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, Math. Inequal. Appl. 18 (2015), no. 2, 443–451.

[7] M. Jleli, L. Ragoub and B. Samet, A Lyapunov-Type Inequality for a Fractional Differential Equation under a Robin Boundary Condition, J. Funct. Spaces, 2015, Art. ID 468536, 5 pp.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[9] A. M. Lyapunov, Probleme général de la stabilité du mouvement, (French Translation of a Russian paper dated 1893), *Ann. Fac. Sci. Univ. Toulouse* 2 (1907), 27–247, Reprinted as Ann. Math. Studies, No, 17, Princeton, 1947.

[10] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993.

[11] Z. Nehari, On the zeros of solutions of second-order linear differential equations, Amer. J. Math. 76 (1954), 689–697.

[12] D. O'Regan and B. Samet, Lyapunov-type inequalities for a class of fractional differential equations, J. Inequal. Appl. 2015, 2015:247.

[13] J. P. Pinasco, *Lyapunov-type inequalities*, Springer Briefs in Mathematics, Springer, New York, 2013.

[14] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999.

[15] J. Rong and C. Bai, Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions, Adv. Difference Equ. 2015, 2015:82, 10 pp.