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# The Existence of Solutions for Classes of Even-Order Differential Equations

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**Abstract**

We establish the existence of multiple positive solutions for an even-order differential equation with right focal boundary conditions. Our technique involves a transformation of the even-order problem into a system of second-order differential equations satisfying homogeneous boundary conditions, as well as multiple applications of the Guo–Krasnosel'skii fixed point theorem to guarantee the existence of at least three positive solutions. We conclude our paper with an existence result for a broader class of the aforementioned second-order system of differential equations.

**AMS Subject Classifications:** 34B18.
**Keywords:** Even order, differential equations, boundary value problem, multiple solutions, positive solutions.

## 1 Introduction

In this paper we continue a current and active study of multiple positive solutions for boundary value problems. Applications of such a study have arisen in beam analysis as shown by Agarwal in [1], which provides an existence and uniqueness result for the fourth-order problem $x^{(4)} = f\left(t, x, x', x'', x^{(3)}\right)$. Moreover, Marcos, Lorca and Ubilla in [5] provided an existence result to the fourth-order differential equation $u^{(4)} = \lambda h\left(t, u(t), u''(t)\right)$ for $t \in (0,1)$ by transforming the fourth-order problem into a

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system of second-order differential equations and then applying the Guo–Krasnosel'skii fixed point theorem. Hopkins then extended this process in [3] by establishing multiple solutions to $u^{(2n)} = \lambda h\left(t, u, u'', \ldots, u^{2(n-1)}\right)$ satisfying right focal boundary conditions.

The particular problem we explore is the even-order differential equation

$$u^{(2n)} = \lambda h\left(t, u, u', \ldots, u^{(2n-1)}\right), \tag{1.1}$$

$$u^{(2k)}(0) = 0, \tag{1.2}$$

$$u^{(2k+1)}(1) = (-1)^k a_k, \tag{1.3}$$

for $t \in (0,1)$, $k = 0, 1, \ldots, n-1$, and where $n \geq 2$; $\lambda, a_k \geq 0$, with $a_0 + a_1 + \ldots + a_{n-1} > 0$, and $h : [0,1] \times [0,\infty)^2 \times (-\infty, 0]^2 \times \cdots \times (-1)^{n-1}[0,\infty)^2 \to (-1)^n[0,\infty)$. In the second section we begin the process of obtaining the existence of multiple positive solutions of (1.1)–(1.3) by transforming the even-order boundary value problem into a system of second-order differential equations satisfying homogeneous boundary conditions. Some preliminaries and conditions that will be needed in order to obtain the required result are also introduced. In the third section we construct a sequence of lemmas that will allow the application of the Guo–Krasnosel'skii fixed point theorem in our main result, Section 4. In Section 5, a corollary will be provided that yields an existence result for a broader class of the aforementioned system of second-order differential equations. This extends the work of Bennett, Brumley, Hopkins, Karber and Milligan in [2].

## 2 Preliminaries

As previously mentioned, we need to transform the even-order differential equation to a second-order system of differential equations. In order to do this, we begin by applying the substitutions

$$\begin{aligned}
u_k &= (-1)^k u^{(2k)}, \quad k = 0, 1, \ldots, n-1, \\
u_{k+1} &= g_k(t, u_0, u_0', u_1, u_1', \ldots, u_{n-1}, u_{n-1}'), \quad k = 0, 1, \ldots, n-2, \\
f(t, &u_0, u_0', u_1, u_1', \ldots, u_{n-1}, u_{n-1}') \\
&= (-1)^n h(t, u_0, u_0', -u_1, -u_1', \ldots, (-1)^{n-1} u_{n-1}, (-1)^{n-1} u_{n-1}'),
\end{aligned}$$

where $t \in (0,1)$. It follows that the solutions of the even-order differential equation (1.1)–(1.3) are in one-to-one correspondence with the solutions of the second-order system

$$-u_{n-1}'' = \lambda f(t, u_0, u_0', \ldots, u_{n-1}, u_{n-1}'), \tag{2.1}$$

$$-u_{n-2}'' = g_{n-2}(t, u_0, u_0', \ldots, u_{n-1}, u_{n-1}'), \tag{2.2}$$

$$\vdots$$

$$-u_0'' = g_0(t, u_0, u_0', \dots, u_{n-1}, u_{n-1}'), \tag{2.3}$$

for $t \in (0,1)$ satisfying boundary conditions

$$u_k(0) = 0, \tag{2.4}$$
$$u_k'(1) = a_k, \tag{2.5}$$

where $k = 0, 1, \dots, n-1$.

Via a simple transformation of (2.1)–(2.5) into a sequence of second-order differential equations, we obtain

$$-u_{n-1}'' = \lambda f(t, u_0 + ta_0, u_0' + a_0, \dots, u_{n-1} + ta_{n-1}, u_{n-1}' + a_{n-1}), \tag{2.6}$$
$$-u_{n-2}'' = g_{n-2}(t, u_0 + ta_0, u_0' + a_0, \dots, u_{n-1} + ta_{n-1}, u_{n-1}' + a_{n-1}), \tag{2.7}$$
$$\vdots$$
$$-u_0'' = g_0(t, u_0 + ta_0, u_0' + a_0, \dots, u_{n-1} + ta_{n-1}, u_{n-1}' + a_{n-1}), \tag{2.8}$$

satisfying the homogeneous boundary conditions,

$$u_k(0) = u_k'(1) = 0, \tag{2.9}$$

for $k = 0, 1, \dots, n-1$. Again notice that once we establish (2.6)–(2.9) has positive solutions, we will be able to conclude that those solutions also satisfy (2.1)–(2.5) as one system is a transformation of the other. Furthermore, in order to obtain our main result, we place the following requirements on the functions $f$ and $g_k$.

(H0) $f : [0,1] \times [0,\infty)^{2n} \to [0,\infty)$ is a continuous function which is nondecreasing in the $2j$th variables and nonincreasing in the $(2j+1)$th variables, where $j = 1, 2, \dots, n$.

(H1) There exists $\alpha, \beta \in (0,1)$, $\alpha < \beta$, such that given $(x_0, x_1, \dots, x_{2n-1}) \in [0,\infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \neq 0$, there exists $k > 0$ such that for $t \in [\alpha, \beta]$,

$$f(t, x_0, x_1, \dots, x_{2n-1}) > k.$$

(H2) Let $z := \sum_{t=0}^{2n-1} x_i$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_0, x_1, \dots, x_{2n-1})}{z} = 0$$

and

$$\lim_{z \to 0^+} \frac{g_k(t, x_0, x_1, \dots, x_{2n-1})}{z} = 0$$

uniformly for $t \in [0,1]$ and $k = 0, 1, \dots, n-2$.

(H3) Let $z := \sum_{t=0}^{2n-1} x_i$. Then

$$\lim_{z\to\infty} \frac{f(t, x_0, x_1, \dots, x_{2n-1})}{z} = 0$$

and

$$\lim_{z\to\infty} \frac{g_k(t, x_0, x_1, \dots, x_{2n-1})}{z} = 0$$

uniformly for $t \in [0, 1]$ and $k = 0, 1, \dots, n-2$.

Now, due to the nature of (2.6)–(2.9), we know any solutions are of the form

$$u_{n-1}(t) = \lambda \int_0^1 G(t,s) f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds,$$

$$u_{n-2}(t) = \int_0^1 G(t,s) g_{n-2}(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds,$$

$$\vdots$$

$$u_0(t) = \int_0^1 G(t,s) g_0(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds,$$

where $G(t, s)$ represents the Green's function

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1. \end{cases}$$

Since $G(t, s)$ is nonnegative and since $f$ and $g_k$ where $k = 0, 1, \dots, n-2$ are nonnegative by assumption, it follows that the above solutions must also be nonnegative. Some other useful properties of $G(t, s)$ are that

$$\max_{t\in[0,1]} \int_0^1 G(t,s)\, ds = \frac{1}{2}$$

and

$$\max_{t\in[0,1]} \int_0^1 \left| \frac{\partial}{\partial t} G(t,s) \right| ds = 1.$$

In our main result, we make use of the Guo–Krasnosel'skii fixed point theorem. Thus, we will need a Banach space, a cone, and an operator $T$. Let $(X, \|\cdot\|)$ denote the Banach space $X = \prod_{j=0}^{n-1} C^1([0, 1]; \mathbb{R})$ endowed with the norm

$$\|(u_0, u_1, \dots, u_{n-1})\| = \sum_{i=0}^{n-1} \left( \|u_i\|_\infty + \|u_i'\|_\infty \right),$$

where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Define $C \subset X$ to be the cone

$$C = \{(u_0, u_1, \dots, u_{n-1}) \in X \mid u_0, u_1, \dots, u_{n-1} \text{ are concave and } (u_0, u_1, \dots, u_{n-1})(0) = (u_0', u_1', \dots, u_{n-1}')(1) = (0, 0, \dots, 0)\}.$$

Notice that $C$ is indeed a cone as it is a nonempty, closed, convex subset of $X$ satisfying both of the following properties:

1. If $x \in C$, and $\lambda > 0$, then $\lambda x \in C$.

2. If $x \in C$ and $-x \in C$, then $x = 0$.

In addition, let $\Omega_p$ denote the open set

$$\Omega_p = \{(u_0, u_1, \dots, u_{n-1}) \in X : \|(u_0, u_1, \dots, u_{n-1})\| < p\}.$$

Lastly, define $T : X \to X$ to be the operator

$$T(u_0, \dots, u_{n-1}) = (A_0(u_0, \dots, u_{n-1}), \dots, A_{n-1}(u_0, \dots, u_{n-1})),$$

where

$$A_{n-1}(u_0, \dots, u_{n-1})(t) = \lambda \int_0^1 G(t, s) f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds$$

and

$$A_k(u_0, \dots, u_{n-1})(t) = \int_0^1 G(t, s) g_k(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) ds$$

for $k = 0, 1, \dots, n-2$.

Notice that the operator $T$ satisfies the following lemma.

**Lemma 2.1.** *$T : C \to C$ is a completely continuous operator.*

One can use a standard Arzela–Ascoli argument to show that $T$ is completely continuous.

We now present the following lemma which will be used in the next section.

**Lemma 2.2.** *Let $u(t)$ be a nonnegative concave function which is continuous on $[0, 1]$. Then for all $\alpha, \beta \in (0, 1)$, with $\alpha < \beta$, we have*

$$\inf_{t\in[\alpha,\beta]} u(t) \geq \alpha(1 - \beta)\|u\|_\infty.$$

This section concludes with the statement of the Guo–Krasnosel'skii fixed point theorem (see [4]) as it will be utilized multiple times to acquire our main result.

**Theorem 2.3.** *Let* $(X, \|\cdot\|)$ *be a Banach space and let* $C \subset X$ *be a cone. Suppose* $\Omega_1, \Omega_2$ *are open subsets of* $X$ *satisfying* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. *If* $T : C \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to C$ *is a completely continuous operator such that either*

*i.* $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_2$, or

*ii.* $\|Tu\| \geq \|u\|$ for $u \in C \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in C \cap \partial\Omega_2$,

*then* $T$ *has a fixed point in* $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

# 3 Technical Results

**Lemma 3.1.** *Suppose* (H0) *and* (H1) *hold and let* $\rho^* > 0$. *Then there exists* $\Lambda > 0$ *such that for all* $\lambda \geq \Lambda$ *and* $(a_0, a_1, \ldots, a_{n-1}) \in [0, \infty)^n$,

$$\|T(u_0, u_1, \ldots, u_{n-1})\| \geq \|(u_0, u_1, \ldots, u_{n-1})\|$$

*for each* $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$.

*Proof.* Let $\rho^* > 0$ and let $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}$. Assume $\alpha$ and $\beta$ are as in (H1) and set $r = \alpha(1-\beta)$. Note that $r \in (0, 1)$. Furthermore, choose $c \geq 1$ where $u_k' + a_k \leq c\|u_k'\|_\infty$ holds for all $k = 0, 1, \ldots, n-1$, and $t \in [\alpha, \beta]$. For $a_0, a_1, \ldots, a_{2n-2} > 0$, $a_{2n-1} \geq 0$, and $a_0 + a_1 + \cdots + a_{2n-1} = \rho^*$, define

$$M = \inf\left\{\frac{f(t, ra_0, ca_1, ra_2, ca_3, \ldots, ra_{2n-2}, ca_{2n-1})}{r(a_0 + a_2 + \cdots + a_{2n-2}) + c(a_1 + a_3 + \cdots + a_{2n-1})} : t \in [\alpha, \beta]\right\}.$$

The existence of a positive $M$ follows from assumption (H1). Set

$$\Lambda \geq \frac{1}{Mr\displaystyle\int_\alpha^\beta G(1, s)ds}.$$

Since $(u_0, u_1, \ldots, u_{n-1}) \in C$, we can utilize (H0) and Lemma 2.2 to obtain

$$\begin{aligned}
&\|T(u_0, u_1, \ldots, u_{n-1})\| \\
&\geq \|A_{n-1}(u_0, u_1, \ldots, u_{n-1})\|_\infty \\
&\geq \lambda \int_0^1 G(1, s) f(s, u_0 + sa_0, u_0' + a_0, \ldots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})ds \\
&\geq \lambda \int_\alpha^\beta G(1, s) f(s, r\|u_0\|_\infty, c\|u_0'\|_\infty, \ldots, r\|u_{n-1}\|_\infty, c\|u_{n-1}'\|_\infty)ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda\big[r\left(\|u_0\|_\infty + \|u_1\|_\infty + \cdots + \|u_{n-1}\|_\infty\right) + c\left(\|u_0'\|_\infty + \|u_1'\|_\infty + \cdots + \|u_{n-1}'\|_\infty\right)\big] \\
&\quad \cdot \int_\alpha^\beta G(1,s)\frac{f(s, r\|u_0\|_\infty, c\|u_0'\|_\infty, \ldots, r\|u_{n-1}\|_\infty, c\|u_{n-1}'\|_\infty)}{r\left(\|u_0\|_\infty + \cdots + \|u_{n-1}\|_\infty\right) + c\left(\|u_0'\|_\infty + \cdots + \|u_{n-1}'\|_\infty\right)}ds \\
&\geq \lambda M\big[r\left(\|u_0\|_\infty + \cdots + \|u_{n-1}\|_\infty\right) + c\left(\|u_0'\|_\infty + \cdots + \|u_{n-1}'\|_\infty\right)\big] \\
&\quad \cdot \int_\alpha^\beta G(1,s)ds \\
&\geq \lambda M r\left(\|u_0\|_\infty + \|u_0'\|_\infty + \cdots + \|u_{n-1}\|_\infty + \|u_{n-1}'\|_\infty\right)\int_\alpha^\beta G(1,s)ds \\
&\geq \lambda M r\|(u_0, u_1, \ldots, u_{n-1})\|\int_\alpha^\beta G(1,s)ds \\
&\geq \Lambda M r\|(u_0, u_1, \ldots, u_{n-1})\|\int_\alpha^\beta G(1,s)ds \\
&\geq \|(u_0, u_1, \ldots, u_{n-1})\|.
\end{aligned}
$$

This concludes the proof. □

**Lemma 3.2.** *Fix* $\Lambda > 0$*. Suppose* (H0) *and* (H1) *hold. Then for all* $\lambda \geq \Lambda$ *and for all* $(a_0, a_1, \ldots, a_{n-1}) \in [0,\infty)^n$ *with* $a_0 + a_1 + \cdots + a_{n-1} > 0$*, there exists* $\rho_1 = \rho_1(\Lambda, a_0, a_1, \ldots, a_{n-1})$ *such that for every* $\rho \in (0, \rho_1]$*, we have*

$$\|T(u_0, u_1, \ldots, u_{n-1})\| \geq \|(u_0, u_1, \ldots, u_{n-1})\|$$

*for all* $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$*.*

*Proof.* Fix $\Lambda > 0$. By (H1) and the nonincreasing/nondecreasing properties of $f$, there exists $k > 0$ such that

$$
\begin{aligned}
f(t, u_0 + ta_0, &u_0' + a_0, \ldots, u_{n-1} + ta_{n-1} + u_{n-1}' + a_{n-1}) \\
&\geq f(t, \alpha a_0, \|u_0'\|_\infty + a_0, \ldots, \alpha a_{n-1}, \|u_{n-1}'\|_\infty + a_{n-1}) \\
&> k
\end{aligned}
$$

for all $t \in (\alpha, \beta)$ where $\alpha$ and $\beta$ are as in (H1). Take $\rho_1 = \Lambda k \int_\alpha^\beta G(1,s)ds$. Then, for $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$ where $\rho \leq \rho_1$,

$$
\begin{aligned}
\|T(u_0, u_1, \ldots, u_{n-1})\| &\geq \|A_2(u_0, u_1, \ldots, u_{n-1})\|_\infty \\
&\geq \lambda \int_0^1 G(1,s) f(s, u_0 + sa_0, u_0' + a_0, \ldots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})ds \\
&\geq \lambda \int_\alpha^\beta G(1,s) f(s, \alpha a_0, \|u_0'\|_\infty + a_0, \ldots, \alpha a_{n-1}, \|u_{n-1}'\|_\infty + a_{n-1})ds
\end{aligned}
$$

$$
\begin{aligned}
&> \lambda k \int_\alpha^\beta G(1,s)ds \\
&= \lambda k \|(u_0, u_1, \ldots, u_{n-1})\| \int_\alpha^\beta \frac{G(1,s)}{\|(u_0, u_1, \ldots, u_{n-1})\|} ds \\
&\geq \Lambda k \|(u_0, u_1, \ldots, u_{n-1})\| \int_\alpha^\beta \frac{G(1,s)}{\|(u_0, u_1, \ldots, u_{n-1})\|} ds \\
&= \frac{\rho_1}{\rho} \|(u_0, u_1, \ldots, u_{n-1})\| \\
&\geq \|(u_0, u_1, \ldots, u_{n-1})\|
\end{aligned}
$$

giving our desired result. □

**Lemma 3.3.** *Suppose* (H0) *and* (H2) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$, *there exists* $\rho_2 \in (0, \rho^*)$ *and* $\delta > 0$ *such that for every* $(a_0, a_1, \ldots, a_{n-1}) \in [0, \infty)^n$, *with* $0 < \sum_{i=0}^{n-1} a_i < \delta$, *we have*

$$
\|T(u_0, u_1, \ldots, u_{n-1})\| \leq \|(u_0, u_1, \ldots, u_{n-1})\|
$$

*for* $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$.

*Proof.* Let $\lambda > 0$, $t \in [0,1]$, and $(a_0, a_1, \ldots, a_{n-1}) \in [0, \infty)^n$ such that $\sum_{i=0}^{n-1} a_i > 0$. Pick $\gamma \in \left(0, \frac{1}{3(\lambda + n - 1)}\right)$. Then, by (H2), we can find $\rho_2 \in (0, \rho^*)$ such that for all $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i = \rho_2$ and $\sum_{i=0}^{n-1} a_i \leq \rho_2$, we have

$$
f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < \gamma \left( \sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i \right)
$$

and

$$
g_k(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < \gamma \left( \sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i \right)
$$

for $k = 0, 1, \ldots, n-2$.

Take $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$, and pick $\delta > 0$ so that $\delta < \rho_2$. Let

$$
c = \min \left\{ c_k \in (0,1] \mid u_k' + a_k \geq c_k \|u_k'\|_\infty, k = 0, 1, \ldots, n-1 \right\}.
$$

Then, for $\sum_{i=0}^{n-1} a_i < \delta$ and by (H0), we have

$$\begin{aligned}
&A_{n-1}(u_0, u_1, \dots, u_{n-1})(t) \\
&\quad = \lambda \int_0^1 G(t,s)\, f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})\, ds \\
&\quad \le \lambda \int_0^1 G(t,s)\, f(s, \|u_0\|_\infty + a_0, c\|u_0'\|_\infty, \dots, \|u_{n-1}\|_\infty + a_{n-1}, c\|u_{n-1}'\|_\infty)\, ds \\
&\quad < \lambda\gamma \sum_{i=0}^{n-1} \left[(\|u_i\|_\infty + a_i) + c\|u_i'\|_\infty\right] \int_0^1 G(t,s)\, ds \\
&\quad \le \lambda\gamma \sum_{i=0}^{n-1} (\|u_i\|_\infty + \|u_i'\|_\infty + a_i) \int_0^1 G(t,s)\, ds \\
&\quad = \lambda\gamma \left[\|(u_0, u_1, \dots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i\right] \int_0^1 G(t,s)\, ds \\
&\quad < 2\lambda\gamma \|(u_0, u_1, \dots, u_{n-1})\| \int_0^1 G(t,s)\, ds \\
&\quad \le \lambda\gamma \|(u_0, u_1, \dots, u_{n-1})\|.
\end{aligned}$$

Using a similar argument to the one above, we see that

$$\begin{aligned}
&A_{n-1}'(u_0, u_1, \dots, u_{n-1})(t) \\
&\quad = \lambda \int_0^1 \frac{\partial}{\partial t} \left[G(t,s)\, f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})\right] ds \\
&\quad < 2\lambda\gamma \|(u_0, u_1, \dots, u_{n-1})\| \int_0^1 \frac{\partial}{\partial t} G(t,s)\, ds \\
&\quad \le 2\lambda\gamma \|(u_0, u_1, \dots, u_{n-1})\|.
\end{aligned}$$

In other words,

$$\|A_{n-1}(u_0, u_1, \dots, u_{n-1})\|_\infty + \|A_{n-1}'(u_0, u_1, \dots, u_{n-1})\|_\infty < 3\lambda\gamma \|(u_0, u_1, \dots, u_{n-1})\|.$$

Due to their construction, each $g_k$, for $k = 0, 1, \dots, n-2$, shares the same nondecreasing and nonincreasing properties as $f$. Therefore, using an argument similar to the one applied above we have that for $i \in \{0, 1, \dots, n-2\}$,

$$\|A_i(u_0, u_1, \dots, u_{n-1})\|_\infty + \|A_i'(u_0, u_1, \dots, u_{n-1})\|_\infty < 3\gamma \|(u_0, u_1, \dots, u_{n-1})\|.$$

Thus, for $\sum_{i=0}^{n-1} a_i < \delta$, we have

$$\|T(u_0, u_1, \dots, u_{n-1})\| = \sum_{i=0}^{n-1} \left[\|A_i(u_0, u_1, \dots, u_{n-1})\|_\infty + \|A_i'(u_0, u_1, \dots, u_{n-1})\|_\infty\right]$$

$$< \left[3\lambda\gamma + \sum_{i=0}^{n-2} 3\gamma\right] \|(u_0, u_1, \ldots, u_{n-1})\|.$$

It follows that $\|T(u_0, u_1, \ldots, u_{n-1})\| \leq \|(u_0, u_1, \ldots, u_{n-1})\|$. □

**Lemma 3.4.** *Let* $\delta > 0$. *Suppose* $0 < \sum_{i=0}^{n-1} a_i < \delta$ *and* (H0) *and* (H3) *hold. Then, for every* $\lambda > 0$*, there exists* $\rho_3 = \rho_3(\delta, \lambda)$ *such that for all* $\rho \geq \rho_3$*,*

$$\|T(u_0, u_1, \ldots, u_{n-1})\| \leq \|(u_0, u_1, \ldots, u_{n-1})\|,$$

*where* $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$.

*Proof.* Let $0 < \sum_{i=0}^{n-1} a_i < \delta$, $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$, and $t \in [0, 1]$. Pick $\eta \in \left(0, \dfrac{1}{3(\lambda + n - 1)}\right)$. By (H0) and (H3), there exists $q_1 > 0$ such that for all $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \geq q_1$, we have

$$f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < \eta \left(\sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i\right)$$

and

$$g_k(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < \eta \left(\sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i\right)$$

for $k = 0, 1, \ldots, n-2$. Let $\rho_3 = \max\{\delta, q_1\}$. Since $\sum_{i=0}^{n-1} a_i < \delta$, for $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \geq \rho_3$, we have

$$\begin{aligned} f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) &< \eta \left[\sum_{i=0}^{2n-1} x_i + \rho_3\right] \\ &< 2\eta \sum_{i=0}^{2n-1} x_i, \end{aligned}$$

and

$$g_k(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < 2\eta \sum_{i=0}^{2n-1} x_i$$

for $k = 0, 1, \ldots, n-2$.

Letting $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_3}$, we can apply a similar argument to the one used in the previous proof to see that

$$\|A_{n-1}(u_0, u_1, \ldots, u_{n-1})\|_\infty + \|A'_{n-1}(u_0, u_1, \ldots, u_{n-1})\|_\infty < 3\lambda\eta\|(u_0, u_1, \ldots, u_{n-1})\|$$

and for $i \in \{0, 1, \ldots, n-2\}$,

$$\|A_i(u_0, u_1, \ldots, u_{n-1})\|_\infty + \|A'_i(u_0, u_1, \ldots, u_{n-1})\|_\infty \le 3\eta\|(u_0, u_1, \ldots, u_{n-1})\|.$$

Thus, for $\rho \ge \rho_3$ and given $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$, we see that

$$\begin{aligned}\|T(u_0, u_1, \ldots, u_{n-1})\| &= \sum_{i=0}^{n-1}\left[\|A_i(u_0, u_1, \ldots, u_{n-1})\|_\infty + \|A'_i(u_0, u_1, \ldots, u_{n-1})\|_\infty\right] \\ &< \left[3\lambda\eta + \sum_{i=0}^{n-2} 3\eta\right]\|(u_0, u_1, \ldots, u_{n-1})\| \\ &< \|(u_0, u_1, \ldots, u_{n-1})\|,\end{aligned}$$

as desired. □

# 4 The Main Result

**Theorem 4.1.** *Let continuous functions* $f, g_k : [0,1] \times [0,\infty)^{2n} \to [0,\infty)$ *satisfy hypotheses* (H0) – (H3). *Then there exists* $\Lambda > 0$ *such that given* $\lambda \ge \Lambda$*, there exists* $\delta > 0$ *such that for every* $a_0, a_1, \ldots, a_{n-1} \ge 0$ *satisfying* $0 < \sum_{i=0}^{n-1} a_i < \delta$*, the system* (2.6)–(2.9) *has at least three positive solutions.*

*Proof.* Suppose $f, g_k$ satisfy hypotheses (H0)–(H3). Let $\rho^* > 0$ be fixed. By Lemma 3.1, there is $\Lambda > 0$ such that, for every $\lambda \ge \Lambda$ and $a_0, a_1, \ldots, a_{n-1} \ge 0$,

$$\|T(u_0, u_1, \ldots, u_{n-1})\| \ge \|(u_0, u_1, \ldots, u_{n-1})\| \text{ for } (u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho^*}.$$

Now, fix $\lambda \ge \Lambda$. Lemmas 3.2 through 3.4 give that there exists $\delta > 0$ and $\rho_1, \rho_2, \rho_3 > 0$ satisfying $\rho_1 < \rho_2 < \rho^* < \rho_3$ such that for $(a_0, a_1, \ldots, a_{n-1}) \in [0,\infty)^n$ with $0 < \sum_{i=0}^{n-1} a_i < \delta$,

$$\begin{aligned}\|T(u_0, u_1, \ldots, u_{n-1})\| &\ge \|(u_0, u_1, \ldots, u_{n-1})\| \text{ for } (u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_1}, \\ \|T(u_0, u_1, \ldots, u_{n-1})\| &\le \|(u_0, u_1, \ldots, u_{n-1})\| \text{ for } (u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}, \\ \|T(u_0, u_1, \ldots, u_{n-1})\| &\le \|(u_0, u_1, \ldots, u_{n-1})\| \text{ for } (u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_3}.\end{aligned}$$

Applying the Guo–Krasnosel'skii fixed point theorem three times, we get the existence of three positive solutions, $(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{n-1}), (\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{n-1}), (\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{n-1}) \in C$ such that

$$\rho_1 < \|(\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{n-1})\| < \rho_2 < \|(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_{n-1})\| \\ < \rho^* < \|(\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{n-1})\| < \rho_3.$$

Thus we see we have at least three positive solutions. □

Recall that solutions to the system (2.6)–(2.9) are in one-to-one correspondence with those of the system (2.1)–(2.5). And the solutions of (2.1)–(2.5) are in one-to-one correspondence with those of our original problem (1.1)–(1.3). Thus we have our desired result.

# 5 Corollary

We now provide an existence result for a broader class of second-order systems of differential equations of the form (2.1)–(2.5) by considering the following modifications to the hypotheses. Note that this system is not derived from an even-order differential equation, thus there is no underlying implicit relationship between $u_0, \ldots, u_{n-1}$ and the corresponding functions, $g_k$. As a result, the hypothesis (H0*) has been amended to incorporate these needed conditions on each $g_k$. We also modify hypotheses (H2) and (H3) by removing conditions related to the functions $g_k$ and replacing those conditions with the looser hypotheses (H4) and (H5).

(H0*) $f, g_k : [0, 1] \times [0, \infty)^{2n} \to [0, \infty)$ for $k = 0, 1, \ldots, n-2$ are continuous functions which are nondecreasing in the $2j$th variables and nonincreasing in the $(2j+1)$th variables, where $j = 1, 2, \ldots, n$.

(H2*) Let $z := \sum_{t=0}^{2n-1} x_i$. Then

$$\lim_{z \to 0^+} \frac{f(t, x_0, x_1, \ldots, x_{2n-1})}{z} = 0$$

uniformly for $t \in [0, 1]$ and $k = 0, 1, \ldots, n-2$.

(H3*) Let $z := \sum_{t=0}^{2n-1} x_i$. Then

$$\lim_{z \to \infty} \frac{f(t, x_0, x_1, \ldots, x_{2n-1})}{z} = 0$$

uniformly for $t \in [0, 1]$ and $k = 0, 1, \ldots, n-2$.

(H4) There exists $\gamma_0, \gamma_1, \ldots, \gamma_{n-2} > 0$ such that $\displaystyle\sum_{k=0}^{n-2} \gamma_k \leq \frac{2}{3}$ and $q > 0$ such that for $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ with $\displaystyle\sum_{i=0}^{2n-1} x_i < q$,

$$g_k(t, x_0, x_1, \ldots, x_{2n-1}) \leq \gamma_k \sum_{i=0}^{2n-1} x_i \text{ for } t \in [0,1], k = 0, 1, \ldots, n-2.$$

(H5) There exists $\eta_0, \eta_1, \ldots, \eta_{n-2} > 0$ such that $\displaystyle\sum_{k=0}^{n-2} \eta_k \leq \frac{2}{3}$ and $\hat{\rho} > 0$ such that for $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ with $\displaystyle\sum_{i=0}^{2n-1} x_i > \hat{\rho}$,

$$g_k(t, x_0, x_1, \ldots, x_{2n-1}) \leq \eta_k \sum_{i=0}^{2n-1} x_i \text{ for } t \in [0,1], k = 0, 1, \ldots, n-2.$$

Just as in the previous problem, we transform the system (2.1)–(2.5) to the problem (2.6)–(2.9) and show the existence of multiple solutions to the transformed system. Note that Lemma 3.1 and Lemma 3.2 still apply, however, we will need to modify the proofs of Lemma 3.3 and Lemma 3.4 to accommodate the new problem.

**Lemma 5.1.** *Suppose* (H0$^*$), (H2$^*$) *and* (H4) *hold, and let* $\rho^* > 0$ *be fixed. Then given* $\lambda > 0$*, there exists* $\rho_2 \in (0, \rho^*)$ *and* $\delta > 0$ *such that for every* $(a_0, a_1, \ldots, a_{n-1}) \in [0, \infty)^n$*, with* $0 < \displaystyle\sum_{i=0}^{n-1} a_i < \delta$*, we have*

$$\|T(u_0, u_1, \ldots, u_{n-1})\| \leq \|(u_0, u_1, \ldots, u_{n-1})\|,$$

*for* $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$.

*Proof.* Let $\lambda > 0$. Pick $\epsilon > 0$ so that $\lambda\epsilon < \dfrac{1}{3}$. Then, by (H2$^*$), we can find a $\rho_2 \in (0, \rho^*)$ such that, for all $(x_0, x_1, \ldots, x_{2n-1}) \in [0, \infty)^{2n}$ such that $\displaystyle\sum_{i=0}^{2n-1} x_i = \rho_2$ and $\displaystyle\sum_{i=0}^{n-1} a_i \leq \rho_2$ with $\rho_2 < \dfrac{1}{2}q$ where $q > 0$ is as in (H4), we have

$$f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) < \epsilon \left( \sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i \right)$$

for $t \in [0,1]$.

Take $(u_0, u_1, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$ and suppose $\sum_{i=0}^{n-1} a_i \le \rho_2$. Set

$$c = \min \left\{ c_k \in (0,1] \mid u_k' + a_k \ge c_k \|u_k'\|_\infty, k = 0, 1, \dots, n-1 \right\}.$$

Using the same reasoning seen in Lemma 3.3, we have that for $t \in [0,1]$,

$$A_{n-1}(u_0, u_1, \dots, u_{n-1})(t) < \lambda\epsilon \|(u_0, u_1, \dots, u_{n-1})\|$$

and

$$A_{n-1}'(u_0, u_1, \dots, u_{n-1})(t) < 2\lambda\epsilon \|(u_0, u_1, \dots, u_{n-1})\|.$$

Thus,

$$\|A_{n-1}(u_0, u_1, \dots, u_{n-1})\|_\infty + \|A_{n-1}'(u_0, u_1, \dots, u_{n-1})\|_\infty \le 3\lambda\epsilon \|(u_0, u_1, \dots, u_{n-1})\|.$$

By hypothesis (H4), since $\sum_{i=0}^{n-1} [(\|u_i\|_\infty + a_i) + \|u_i'\|_\infty] \le 2\rho_2 < q$, then there is $\gamma_0, \gamma_1, \dots, \gamma_{n-2} > 0$, such that $\sum_{k=0}^{n-2} \gamma_k \le \frac{2}{3}$ and such that

$$g_k(t, \|u_0\|_\infty + a_0, \|u_0'\|_\infty, \dots, \|u_{n-1}\|_\infty + a_{n-1}, \|u_{n-1}'\|_\infty) \le \gamma_k \sum_{i=0}^{n-1} [(\|u_i\|_\infty + a_i) + \|u_i'\|_\infty].$$

Let $\delta' < 1$ and set $\delta = \delta' \rho_2$. Then for $\sum_{i=0}^{n-1} a_i < \delta$, $(u_0, u_1, \dots, u_{n-1}) \in C \cap \partial\Omega_{\rho_2}$, and $t \in [0,1]$, we have

$$\begin{aligned}
&A_k(u_0, u_1, \dots, u_{n-1})(t) \\
&\quad = \int_0^1 G(t,s)\, g_k(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})\, ds \\
&\quad \le \int_0^1 G(t,s)\, g_k(s, \|u\|_\infty + a_0, c\|u'\|_\infty, \dots, \|u_{n-1}\|_\infty + a_{n-1}, c\|u_{n-1}'\|_\infty)\, ds \\
&\quad \le \gamma_k \left[ \|u_0\|_\infty + c\|u_0'\|_\infty + \cdots + \|u_{n-1}\|_\infty + c\|u_{n-1}'\|_\infty + \sum_{i=0}^{n-1} a_i \right] \int_0^1 G(t,s)\, ds \\
&\quad \le \gamma_k \left[ \|(u_0, u_1, \dots, u_{n-1})\| + \sum_{i=0}^{n-1} a_i \right] \int_0^1 G(t,s)\, ds
\end{aligned}$$

$$< \gamma_k(1+\delta')\|(u_0,u_1,\dots,u_{n-1})\| \int_0^1 G(t,s)\,ds$$
$$\le \frac{1}{2}\gamma_k(1+\delta')\|(u_0,u_1,\dots,u_{n-1})\|$$

and

$$A_k'(u_0,u_1,\dots,u_{n-1})(t)$$
$$= \int_0^1 \frac{\partial}{\partial t}\left[G(t,s)g_k(s,u_0+sa_0,u_0'+a_0,\dots,u_{n-1}+sa_{n-1},u_{n-1}'+a_{n-1})\right]\,ds$$
$$< \gamma_k(1+\delta')\|(u_0,u_1,\dots,u_{n-1})\| \int_0^1 \frac{\partial}{\partial t}G(t,s)\,ds$$
$$\le \gamma_k(1+\delta')\|(u_0,u_1,\dots,u_{n-1})\|$$

for $k=0,1,\dots,n-2$ and $c$ is as above. Hence,

$$\|A_k(u_0,\dots,u_{n-1})\|_\infty + \|A_k'(u_0,\dots,u_{n-1})\|_\infty < \frac{3}{2}\gamma_k(1+\delta')\|(u_0,\dots,u_{n-1})\|$$

for $k=0,1,\dots,n-2$.

Thus, for $\sum_{i=0}^{n-1} a_i < \delta$, we have

$$\|T(u_0,u_1,\dots,u_{n-1})\| = \sum_{k=0}^{n-1}\left[\|A_k(u_0,u_1,\dots,u_{n-1})\|_\infty + \|A_k'(u_0,u_1,\dots,u_{n-1})\|_\infty\right]$$
$$< \left[\sum_{k=0}^{n-2}\frac{3}{2}\gamma_k(1+\delta') + 3\lambda\epsilon\right]\|(u_0,u_1,\dots,u_{n-1})\|.$$

For small enough $\epsilon$ and $\delta'$, it follows that $\|T(u_0,u_1,\dots,u_{n-1})\| \le \|(u_0,u_1,\dots,u_{n-1})\|$. □

**Lemma 5.2.** *Let* $\delta > 0$. *Suppose* $0 < \sum_{i=0}^{n-1} a_i < \delta$ *and* (H0*), (H3*), *and* (H5) *hold. Then, for every* $\lambda > 0$, *there exists* $\rho_3 = \rho_3(\delta,\lambda)$ *such that for all* $\rho \ge \rho_3$,

$$\|T(u_0,u_1,\dots,u_{n-1})\| \le \|(u_0,u_1,\dots,u_{n-1})\|,$$

*where* $(u_0,u_1,\dots,u_{n-1}) \in C \cap \partial\Omega_\rho$.

*Proof.* Let $0 < \sum_{i=0}^{n-1} a_i < \delta$ and let $(x_0,x_1,\dots,x_{2n-1}) \in [0,\infty)^{2n}$. By (H5) and by the nondecreasing/nonincreasing properties of $g_k$ in (H0*) for $k=0,1,\dots,n-2$, given any $q_1 \ge \hat{\rho}$, we have

$$g_k(t,x_0+a_0,x_1,x_2+a_1,x_3,\dots,x_{2n-2}+a_{n-1},x_{2n-1}) \le \eta_k\left(\sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i\right)$$

for $\sum_{i=0}^{2n-1} x_i \geq q_1$ and $t \in [0,1]$, where $\sum_{k=0}^{n-2} \eta_k \leq \frac{2}{3}$.

Let $\epsilon > 0$ and $\eta = \max\{\eta_0, \ldots, \eta_{n-2}\}$. Pick $q_1 \geq \hat{\rho}$ large enough so that $\epsilon > \frac{\eta\delta}{q_1}$. Let $\sum_{i=0}^{2n-1} x_i \geq q_1$. Then

$$\begin{aligned} g_k(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \ldots, x_{2n-2} + a_{n-1}, x_{2n-1}) &\leq \eta_k \left( \sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i \right) \\ &< (\eta_k + \epsilon) \sum_{i=0}^{2n-1} x_i. \end{aligned}$$

Let $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_{q_1}$. Let

$$c = \min \left\{ c_k \in (0,1] \mid u_k' + a_k \geq c_k \|u_k'\|_\infty, k = 0, 1, \ldots, n-1 \right\}.$$

Then for $t \in [0,1]$ and $k = 0, 1, \ldots, n-2$,

$$\begin{aligned} &A_k(u_0, u_1, \ldots, u_{n-1})(t) \\ &\quad = \int_0^1 G(t,s)\, g_k(s, u_0 + s a_0, u_0' + a_0, \ldots, u_{n-1} + s a_{n-1}, u_{n-1}' + a_{n-1})\, ds \\ &\quad \leq \int_0^1 G(t,s)\, g_k(s, \|u_0\|_\infty + a_0, c\|u_0'\|_\infty, \ldots, \|u_{n-1}\|_\infty + a_{n-1}, c\|u_{n-1}'\|_\infty)\, ds \\ &\quad < (\eta_k + \epsilon)\, \|(u_0, u_1, \ldots, u_{n-1})\| \int_0^1 G(t,s)\, ds \end{aligned}$$

and

$$\begin{aligned} &A_k'(u_0, u_1, \ldots, u_{n-1})(t) \\ &\quad = \int_0^1 \frac{\partial}{\partial t} \left[ G(t,s)\, g_k(s, u_0 + s a_0, u_0' + a_0, \ldots, u_{n-1} + s a_{n-1}, u_{n-1}' + a_{n-1}) \right] ds \\ &\quad < (\eta_k + \epsilon)\, \|(u_0, u_1, \ldots, u_{n-1})\| \int_0^1 \frac{\partial}{\partial t} G(t,s)\, ds. \end{aligned}$$

Combining these inequalities, we see that

$$\|A_k(u_0, u_1, \ldots, u_{n-1})\|_\infty + \|A_k'(u_0, u_1, \ldots, u_{n-1})\|_\infty < \frac{3(\eta_k + \epsilon)}{2} \|(u_0, u_1, \ldots, u_{n-1})\|$$

for $k = 0, 1, \ldots, n-2$.

Now consider $A_{n-1}(u_0, u_1, \dots, u_{n-1})(t)$. Let $\delta' > 0$. Then, by (H0*) and (H3*), there is a $q_2 > 0$ such that for all $(x_0, x_1, \dots, x_{2n-1}) \in [0,\infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \geq q_2$, we have

$$f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \dots, x_{2n-2} + a_{n-1}, x_{2n-1}) \leq \delta' \left( \sum_{i=0}^{2n-1} x_i + \sum_{i=0}^{n-1} a_i \right)$$

for every $t \in [0,1]$. Let $q_3 = \max\{\delta, q_2\}$. Note that $\sum_{i=0}^{n-1} a_i < \delta$, for $(x_0, \dots, x_{n-1}) \in [0,\infty)^{2n}$ with $\sum_{i=0}^{2n-1} x_i \geq q_3$, we see that

$$\begin{aligned} f(t, x_0 + a_0, x_1, x_2 + a_1, x_3, \dots, x_{2n-2} + a_{n-1}, x_{2n-1}) &\leq \delta' \left[ \sum_{i=0}^{2n-1} x_i + q_3 \right] \\ &\leq 2\delta' \sum_{i=0}^{2n-1} x_i. \end{aligned}$$

Then for $t \in [0,1]$ and any $(u_0, u_1, \dots, u_{n-1}) \in C \cap \partial\Omega_{q_3}$,

$$\begin{aligned} &A_{n-1}(u_0, u_1, \dots, u_{n-1})(t) \\ &\quad = \lambda \int_0^1 G(t,s)\, f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1})\, ds \\ &\quad \leq \lambda \int_0^1 G(t,s)\, f(s, \|u\|_\infty + a_0, c\|u'\|_\infty, \dots, \|u_{n-1}\|_\infty + a_{n-1}, c\|u_{n-1}'\|_\infty)\, ds \\ &\quad < \lambda \cdot 2\delta' \, \|(u_0, u_1, \dots, u_{n-1})\| \int_0^1 G(t,s)\, ds \\ &\quad \leq \lambda\delta' \, \|(u_0, u_1, \dots, u_{n-1})\|, \end{aligned}$$

where $c$ is as above. And similarly,

$$\begin{aligned} &A_{n-1}'(u_0, u_1, \dots, u_{n-1})(t) \\ &\quad = \lambda \int_0^1 \frac{\partial}{\partial t} \left[ G(t,s)\, f(s, u_0 + sa_0, u_0' + a_0, \dots, u_{n-1} + sa_{n-1}, u_{n-1}' + a_{n-1}) \right] ds \\ &\quad < \lambda \cdot 2\delta' \, \|(u_0, u_1, \dots, u_{n-1})\| \int_0^1 \frac{\partial}{\partial t} G(t,s)\, ds \\ &\quad \leq 2\lambda\delta' \, \|(u_0, u_1, \dots, u_{n-1})\|. \end{aligned}$$

Combining these inequalities, we have

$$\|A_{n-1}(u_0, u_1, \dots, u_{n-1})\|_\infty + \|A_{n-1}'(u_0, u_1, \dots, u_{n-1})\|_\infty < 3\lambda\delta' \|(u_0, u_1, \dots, u_{n-1})\|.$$

Take $\rho_3 = \max\{q_1, q_3\}$ and let $\rho \geq \rho_3$. Then given $(u_0, u_1, \ldots, u_{n-1}) \in C \cap \partial\Omega_\rho$, it follows that

$$\begin{aligned}\|T(u_0, u_1, \ldots, u_{n-1})\| &= \sum_{k=0}^{n-1} [\|A_k(u_0, u_1, \ldots, u_{n-1})\|_\infty + \|A'_k(u_0, u_1, \ldots, u_{n-1})\|_\infty] \\ &< \left[\sum_{k=0}^{n-2} \frac{3}{2}(\eta_k + \epsilon) + 3\lambda\delta'\right] \|(u_0, u_1, \ldots, u_{n-1})\|.\end{aligned}$$

Recall by (H5) that $\sum_{k=0}^{n-2} \eta_k \leq \frac{2}{3}$. Picking $\epsilon$ and $\delta'$ small enough yields the desired result. □

**Corollary 5.3.** *Let continuous functions* $f, g_k : [0,1] \times [0,\infty)^{2n} \to [0,\infty)$ *for* $k = 0, 1, \ldots, n-2$ *satisfy hypotheses* (H0*) – (H5). *Then there exists* $\Lambda > 0$ *such that given* $\lambda \geq \Lambda$*, there exists* $\delta > 0$ *such that for every* $a_0, a_1, \ldots, a_{n-1} \geq 0$ *satisfying* $0 < \sum_{i=0}^{n-1} a_i < \delta$*, the system* (2.6)–(2.9) *has at least three positive solutions.*

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