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Boundary Value Problems for Linear Singularly Perturbed Discrete Systems

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Abstract

In this paper, we study boundary value problems for a class of linear singularly perturbed discrete systems. We give conditions ensuring the existence and uniqueness of the solution and a convergent iterative algorithm to compute uniform asymptotic solutions. We use the natural perturbation method, the stiffness of the system is removed and the boundary value problems are switched to plain initial subsystems of reduced order. This method improves the *singular perturbation method* known for this kind of systems.

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1 Introduction

Many engineering problems are described by discrete time systems of high dimension involving small parameters. Therefore, it is desirable to develop reduced-order models and remove the stiffness. In this respect, the *singular perturbation method* has been used to study this kind of systems. This procedure is extensively used for continuous-time systems, see [2, 12, 13], and similar ideas based on boundary layer correction terms have been adopted for autonomous discrete-time systems, see [1, 3–6, 8, 9]. In [11, 14], we confirmed that there is no need for the boundary layer correction terms, and we described a uniform perturbation method we extended later in [15, 16] for some kinds

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of singularly perturbed systems. In this paper we continue our investigation and we describe the method for the C-model:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11}(k) & \varepsilon A_{12}(k) \\ A_{21}(k) & \varepsilon A_{22}(k) \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad k = 0, \ldots, N-1, \tag{1.1}$$

and the R-Model:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ \varepsilon A_{21}(k) & \varepsilon A_{22}(k) \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad k = 0, \ldots, N-1. \tag{1.2}$$

For both systems (1.1) and (1.2), we associate the boundary values

$$y_0 = \alpha, \quad x_N = \beta, \tag{1.3}$$

where α and β are given vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. It is always pointed to study boundary value problems which are often encountered in optimal control, see [10]. The structure of the paper is as follows. In Section 2, we study the boundary value problem (1.1)–(1.3). We give conditions ensuring existence and uniqueness of the solution and we develop the method for computing uniform asymptotic approximate solutions. An algorithm is provided to show the chain of steps of this process. Section 3 is devoted to the R-model, we give similar results for the boundary value problem (1.2)–(1.3). We end this paper with a short conclusion in Section 4.

2 Boundary Value Problem for the C-Model

In this section we study the boundary value problem (1.1)–(1.3) and the main result is given in Theorem 2.1. The particular case of this problem, where the system matrix does not depend on the discrete time k, i.e., the autonomous case, is studied in [9] and the authors have use the *singular perturbation method*.

2.1 Main Result

To describe our *perturbation method* for problem (1.1)–(1.3), we write the solution $(x_k(\varepsilon), y_k(\varepsilon))', k = 0, \ldots, N$, as a straightforward development:

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \ldots, N. \tag{2.1}$$

We substitute the series (2.1) in (1.1)–(1.3). Equating the terms of same order, result the following equations. For the zero*th* order approximation,

$$x_{k+1}^{(0)} = A_{11}(k) x_k^{(0)}, \quad k = 0, \ldots, N-1, \quad x_N^{(0)} = \beta, \tag{2.2}$$

$$y_0^{(0)} = \alpha, \tag{2.3}$$

$$y_{k+1}^{(0)} = A_{21}(k)x_k^{(0)}, \quad k = 0, \ldots, N-1. \tag{2.4}$$

The system (2.2)–(2.3)–(2.4) defines the *reduced problem* of the original problem (1.1)–(1.3). Actually it is composed of the final value problem (2.2) and the algebraic equations (2.4). We notice that all the coefficients $x_k^{(0)}$, $k = 0, \ldots, N-1$, and $y_k^{(0)}$, $k = 1, \ldots, N$, can be computed without any knowledge of the initial condition (2.3). By connection with the differential equations, we say that there is boundary layer at y_0. If the matrices $A_{11}(k)$, $k = 0, \ldots, N-1$, are nonsingular, the trivial solution of the final value problem (2.2) is

$$x_k^{(0)} = \prod_{i=k}^{N-1} A_{11}^{-1}(i)\beta, \quad k = N-1, N-2 \ldots, 0, \quad x_N^{(0)} = \beta. \tag{2.5}$$

Therefore, from (2.3), (2.4) and (2.5), we have

$$y_0^{(0)} = \alpha, \quad y_{k+1}^{(0)} = A_{21}(k) \prod_{i=k}^{N-1} A_{11}^{-1}(i)\beta, \quad k = 0, \ldots, N-1. \tag{2.6}$$

For higher order approximation j, $j \geq 1$, we find

$$x_{k+1}^{(j)} = A_{11}(k)x_k^{(j)} + A_{12}(k)y_k^{(j-1)}, \quad k = 0, \ldots, N-1, \tag{2.7}$$

$$x_N^{(j)} = 0, \tag{2.8}$$

$$y_0^{(j)} = 0, \tag{2.9}$$

$$y_{k+1}^{(j)} = A_{21}(k)x_k^{(j)} + A_{22}(k)y_k^{(j-1)}, \quad k = 0, \ldots, N-1. \tag{2.10}$$

If the matrices $A_{11}(k)$, $k = 0, \ldots, N-1$, are nonsingular, we can compute the coefficients $x_k^{(j)}$, $k = 0, \ldots, N$, recursively with the final value (2.8) from

$$x_k^{(j)} = A_{11}^{-1}(k)x_{k+1}^{(j)} - A_{11}^{-1}(k)A_{12}(k)y_k^{(j-1)}, \quad k = 0, \ldots, N-1, \tag{2.11}$$

where the terms $y_k^{(j-1)}$, $k = 0, \ldots, N-1$, are determined from the previous development of order $j-1$. The terms $y_k^{(j)}$, $k = 1, \ldots, N$, are computed from (2.10) regardless of the value (2.9) but need terms from the previous steps. For all subsystems defined above, the time-scale is separated and their dimensions are equal to n or m, lower than the dimension of the original system equal to $n+m$. To validate the approximations defined above, let

$$v = (x_0, y_0, x_1, y_1, \ldots, x_N, y_N)', \tag{2.12}$$

where the prime denotes the transpose. We consider for v the norm in $\mathbb{R}^{(n+m)(N+1)}$

$$\|v\| = \max\left(|x_0|, |y_0|, |x_1|, |y_1|, \ldots, |x_N|, |y_N|\right),$$

and for a matrix $A = (a_{ij})$, we associate the matrix norm

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \max_{k=0,\dots,(n+m)(N+1)} \left(\sum_{j=0}^{(n+m)(N+1)} |a_{ij}| \right).$$

Theorem 2.1. *If the matrices $A_{11}(k)$, $k = 0, \dots, N-1$, are nonsingular, there exists a positive real number ε_0, for all ε such that $|\varepsilon| < \varepsilon_0$, the solution $(x_k(\varepsilon), y_k(\varepsilon))'$, $0 \le k \le N$, of problem* (1.1)–(1.3)*, satisfies* (2.1) *uniformly for $0 \le k \le N$, where the terms $x_k^{(0)}$, $y_k^{(0)}$, and $x_k^{(j)}$, $y_k^{(j)}$, $j \ge 1$, for $0 \le k \le N$, are respectively the solutions of* (2.5)*,* (2.6)*, and* (2.11)*,* (2.8)*,* (2.9)*,* (2.10)*. Moreover for all $k = 0, \dots, N$, we have*

$$\begin{aligned} \left| x_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j x_k^{(j)} \right| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \\ \left| y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)} \right| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \end{aligned} \tag{2.13}$$

where C is a positive constant.

Proof. We represent the linear system (2.2)–(2.3)–(2.4) in the matrix form

$$A_0 v^{(0)} = f, \tag{2.14}$$

where $v^{(0)}$ and f are vectors in $\mathbb{R}^{(n+m)(N+1)}$ defined by

$$v^{(0)} := \left(x_0^{(0)}, y_0^{(0)}, x_1^{(0)}, y_1^{(0)}, \dots, x_N^{(0)}, y_N^{(0)} \right)', \tag{2.15}$$

$$f := (\alpha, 0, 0, \dots, 0, \beta)', \tag{2.16}$$

and A_0 is the following block matrix

$$\begin{pmatrix} 0 & I_m & 0 & 0 & \dots & & & & 0 \\ A_{11}(0) & 0 & -I_n & 0 & & & & & \\ A_{21}(0) & 0 & 0 & -I_m & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & A_{11}(N-1) & 0 & -I_n & 0 \\ & & & & & A_{21}(N-1) & 0 & 0 & -I_m \\ 0 & & & & \dots & 0 & 0 & I_n & 0 \end{pmatrix}.$$

We find by the Leibniz formula [7] for determinant of block matrices,

$$\det A_0 = \prod_{k=0}^{N-1} \det A_{11}(k).$$

Obviously A_0 is nonsingular if the matrices $A_{11}(k)$, $k = 0, \ldots, N-1$, are nonsingular. We can denote

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|}, \qquad C := \|A_0^{-1}\|\|f\|. \tag{2.17}$$

The system (2.7)–(2.8)–(2.9)–(2.10) may be represented in the form

$$A_0 v^{(j)} = -Uv^{(j-1)}; \quad v^{(j)} := \left(x_0^{(j)}, y_0^{(j)}, x_1^{(j)}, y_1^{(j)}, \ldots, x_N^{(j)}, y_N^{(j)}\right)', \tag{2.18}$$

where U is the matrix given below

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & & & & 0 \\ 0 & A_{12}(0) & 0 & 0 & & & & & \\ 0 & A_{22}(0) & 0 & 0 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & 0 & A_{12}(N-1) & 0 & 0 \\ & & & & & 0 & A_{22}(N-1) & 0 & 0 \\ 0 & & & & \ldots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The problem defined by (1.1)–(1.3) is written in the matrix form

$$A_\varepsilon v = f, \tag{2.19}$$

with v and f are defined by (2.12) and (2.16) respectively, and A_ε satisfies

$$A_\varepsilon = A_0 + \varepsilon U.$$

As $|\varepsilon| < \varepsilon_0$, from (2.17) we have $\|\varepsilon U A_0^{-1}\| < 1$. Thus we can write

$$A_0^{-1} \sum_{j=0}^{\infty} \left(-\varepsilon U A_0^{-1}\right)^j = A_0^{-1} \left(I + \varepsilon U A_0^{-1}\right)^{-1} = A_\varepsilon^{-1}. \tag{2.20}$$

Consequently, the solution of system (2.19) exists, is unique and satisfies

$$v(\varepsilon) = A_\varepsilon^{-1} f. \tag{2.21}$$

From (2.20) and (2.21), we have

$$v(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{(j)} v^{(j)}; \quad v^{(j)} = A_0^{-1} \left(-UA_0^{-1}\right)^j f. \tag{2.22}$$

For $j \geq 1$, from (2.14), (2.15), (2.16) and (2.18), it is clear that the components $x_k^{(0)}$,$y_k^{(0)}$, and $x_k^{(j)}$, $y_k^{(j)}$ are the solutions of the problems (2.2), (2.3), (2.4) and (2.7), (2.8), (2.9), (2.10), respectively, and the equations (2.2) and (2.3)−(2.4), (2.7) are respectively equivalent to (2.5), (2.6), and (2.11).

For the remainder of the series (2.20), we have

$$\begin{aligned}&\left\|A_\varepsilon^{-1}-A_0^{-1}\sum_{j=0}^{n}\left(-\varepsilon UA_0^{-1}\right)^{(j)}\right\|\leq\|A_0^{-1}\|\sum_{j=n+1}^{\infty}\|\varepsilon UA_0^{-1}\|^j\\&=\frac{\|A_0^{-1}\|\|\varepsilon UA_0^{-1}\|^{n+1}}{1-\|\varepsilon UA_0^{-1}\|}\leq\|A_0^{-1}\|\frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1-|\varepsilon|/\varepsilon_0}.\end{aligned}\tag{2.23}$$

Therefore, from (2.20), (2.22), and (2.23), we have

$$\begin{aligned}&\left\|v(\varepsilon)-\sum_{j=0}^{n}\varepsilon^jv^{(j)}\right\|\leq\left\|A_\varepsilon^{-1}-A_0^{-1}\sum_{j=0}^{n}\left(-\varepsilon UA_0^{-1}\right)^j\right\|\|f\|\\&\leq\|A_0^{-1}\|\|f\|\frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1-|\varepsilon|/\varepsilon_0},\end{aligned}\tag{2.24}$$

and from (2.17), we find (2.13). This completes the proof. □

2.2 Algorithm

We give a recursive convergent algorithm for computing approximate solutions for the problem (1.1)–(1.3).

***Zeroth*-order solution**

- Step 1. Fix $x_N^{(0)}=\beta$, and solve (2.5), what gives $x_k^{(0)}$, for $k=0,\ldots,N-1$.
- Step 2. Fix $y_0^{(0)}=\alpha$, and compute $y_k^{(0)}$ for $k=1,\ldots,N$, from (2.6).

***First*-order solution**

- Step 3. Fix $x_N^{(1)}=0$, and compute $x_k^{(1)}$ from (2.11), for $k=0,\ldots,N-1$, where $y_k^{(0)}$, $k=0,\ldots,N-1$, are determined in step 2.
- Step 4. Fix $y_0^{(1)}=0$, and compute $y_k^{(1)}$ from (2.10), for $k=1,\ldots,N$, where for $k=0,\ldots,N-1$. Notice that $x_k^{(1)}$ are determined in step 3 and $y_k^{(0)}$ are determined in step 2.

***J*th-order solution**

- Step 5. Fix $x_N^{(j)}=0$, and compute $x_k^{(j)}$ from (2.11), for $k=0,\ldots,N-1$, where $y_k^{(j-1)}$, $k=0,\ldots,N-1$, are determined in a previous step.
- Step 6. Fix $y_0^{(j)}=0$, and compute $y_k^{(j)}$ from (2.10), for $k=1,\ldots,N$, where for $k=0,\ldots,N-1$: $x_k^{(j)}$ and $y_k^{(j-1)}$ are determined in previous steps.
  Then determine for all $0\leq k\leq N$, the approximate solution of order j:
  $\left(x_k^{(0)},y_k^{(0)}\right)'+\varepsilon\left(x_k^{(1)},y_k^{(1)}\right)'+\ldots+\varepsilon^j\left(x_k^{(j)},y_k^{(j)}\right)'$.

3 Boundary Value Problem for the R-Model

In this section, we focus on the R-model. We describe the *perturbation method* for the boundary value problem (1.2)–(1.3).

3.1 Main Result

Evermore the solution $(x_k(\varepsilon), y_k(\varepsilon))'$, $0 \le k \le N$, of problem (1.2)–(1.3), is written in the form:

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \dots, N. \tag{3.1}$$

Therefore the coefficients of ε^0 in the series above satisfy the following final value problem

$$x_{k+1}^{(0)} = A_{11}(k) x_k^{(0)} + A_{12}(k) y_k^{(0)}, \quad k = 0, \dots, N-1, \tag{3.2}$$

$$x_N^{(0)} = \beta, \tag{3.3}$$

and

$$y_0^{(0)} = \alpha, \quad y_k^{(0)} = 0, \quad k = 1, \dots, N. \tag{3.4}$$

If the matrices $A_{11}(k)$, $k = 0, \dots, N-1$, are nonsingular, the trivial solution of the final value problem (3.2)–(3.3) is

$$\begin{aligned} x_0^{(0)} &= \prod_{i=0}^{N-1} A_{11}^{-1}(i)\beta - A_{11}^{-1}(0) A_{12}(0)\alpha \\ x_k^{(0)} &= \prod_{i=k}^{N-1} A_{11}^{-1}(i)\beta, \quad k = 1, \dots, N-1, \\ x_N^{(0)} &= \beta. \end{aligned} \tag{3.5}$$

The coefficients of ε^j, $j \ge 1$, satisfy the following equations

$$x_{k+1}^{(j)} = A_{11}(k) x_k^{(j)} + A_{12}(k) y_k^{(j)}, \quad k = 0, \dots, N-1, \tag{3.6}$$

$$x_N^{(j)} = 0, \tag{3.7}$$

and

$$y_0^{(j)} = 0, \tag{3.8}$$

$$y_{k+1}^{(j)} = A_{21}(k) x_k^{(j-1)} + A_{22}(k) y_k^{(j-1)}, \quad k = 0, \dots, N-1. \tag{3.9}$$

The system (3.6)–(3.7) is a final value problem. If the matrices $A_{11}(k)$, $k = 0, \dots, N-1$, are nonsingular, then the terms $x_k^{(j)}$, $k = 1, \dots, N-1$ are recursively computed with the final value (3.7) from the difference equation

$$\begin{aligned} &x_k^{(j)} = A_{11}^{-1}(k) \left[x_{k+1}^{(j)} - A_{12}(k) \left(A_{21}(k-1) x_{k-1}^{(j-1)} + A_{22}(k-1) y_{k-1}^{(j-1)} \right) \right], \\ &k = 1, \dots, N-1, \end{aligned} \tag{3.10}$$

and

$$x_0^{(j)} = A_{11}^{-1}(0)x_1^{(j)}. \tag{3.11}$$

The convergence of the development above is established in the following theorem.

Theorem 3.1. *There exists a positive real number ε_0, for all ε such that $|\varepsilon| < \varepsilon_0$, the solution $(x_k(\varepsilon), y_k(\varepsilon))'$, $0 \le k \le N$, of problem* (1.2)–(1.3)*, satisfies* (3.1) *uniformly for $0 \le k \le N$, where $x_k^{(0)}$, $y_k^{(0)}$, $x_k^{(j)}$, and $y_k^{(j)}$ are the solutions of* (3.5)*,* (3.4)*, and* (3.11)*,* (3.10)–(3.7)*,* (3.8)–(3.9)*, respectively. Moreover, for all $0 \le k \le N$, we have*

$$\begin{aligned} \left| x_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j x_k^{(j)} \right| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \\ \left| y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)} \right| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \end{aligned} \tag{3.12}$$

where ε and C are given by (3.13)*.*

Proof. The proof is similar to that of Theorem 2.1. Briefly, we write the problem (1.2)–(1.3) in the matrix form

$$A_\varepsilon v = f,$$

where

$$v = (x_0, y_0, x_1, y_1, \ldots, x_N, y_N)',$$
$$f = (\alpha, 0, 0, \ldots, 0, \beta)'.$$

The matrix A_ε is the combination $A_0 + \varepsilon U$, where A_0 is the following block matrix

$$\begin{pmatrix} 0 & I_m & 0 & 0 & \ldots & & & & 0 \\ A_{11}(0) & A_{12}(0) & -I_n & 0 & & & & & \\ 0 & 0 & 0 & -I_m & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & A_{11}(N-1) & A_{12}(N-1) & -I_n & 0 \\ & & & & & 0 & 0 & 0 & -I_m \\ 0 & & & & \ldots & 0 & 0 & I_n & 0 \end{pmatrix},$$

and U the matrix given below

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & & & & 0 \\ 0 & 0 & 0 & 0 & & & & & \\ A_{21} & A_{22}(0) & 0 & 0 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & A_{21}(N-1) & A_{22}(N-1) & 0 & 0 \\ 0 & & & & \ldots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because the matrix A_0 given above is nonsingular, we denote

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|}, \qquad C := \|A_0^{-1}\|\|f\|. \tag{3.13}$$

The following of the proof is routine and left to the reader. □

3.1.1 Algorithm

***Zeroth*-order approximation**

- Step 1. Fix $x_N^{(0)} = \beta$, compute $x_k^{(0)}$, $1 \leq k \leq N$, from (3.5). Define $y_k^{(0)}$, $0 \leq k \leq N$ from (3.4).

***J*th-order approximation**

- Step 2. Fix $y_0^{(j)} = 0$, and compute $y_k^{(j)}$, $0 \leq k \leq N$, from (3.9), where $x_k^{(j-1)}$ and $y_k^{(j-1)}$ are determined from the previous step.

- Step 3. Fix $x_N^{(j)} = 0$, and compute $x_k^{(j)}$, $0 \leq k \leq N-1$, from (3.10), (3.11), where $x_{k-1}^{(j-1)}$ and $y_{k-1}^{(j-1)}$ are determined from the previous step.
  Hence the approximate solution of order j for $k = 0, \ldots, N$ is:

$$\left(x_k^{(0)}, y_k^{(0)}\right)' + \varepsilon \left(x_k^{(1)}, y_k^{(1)}\right)' + \ldots + \varepsilon^j \left(x_k^{(j)}, y_k^{(j)}\right)'.$$

4 Conclusion

We have considered boundary value problems for a class of time-varying discrete singularly perturbed systems said *slow sampling rate models* or C and R models. The solution of boundary value problems is always a matter. For both models, we find convergent recursive algorithms for computing approximate asymptotic solutions. Besides the advantage of removing the time scale and the decomposition of the full system in subsystems of reduced order with decoupled state variables there is no need of *boundary layer correction* terms. Moreover, We give conditions ensuring the existence and uniqueness of the solution. We can extend this method for boundary values defined as linear combinations of other terms of the sequence, and we can consider final value problems or initial value problems.

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