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Maximum Number and Distribution of Limit Cycles in the General Liénard Polynomial System

Valery A. Gaiko
National Academy of Sciences of Belarus
United Institute of Informatics Problems
Surganov Str. 6, Minsk 220012, Belarus
valery.gaiko@gmail.com

Abstract

In this paper, using our bifurcational geometric approach, we complete the solution of the problem on the maximum number and distribution of limit cycles in the general Liénard polynomial system.

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1 Introduction

In this paper, we continue studying the Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0 \tag{1.1}$$

and the corresponding dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y \tag{1.2}$$

which we have done in [7, 10, 11, 13, 14, 16–20].

We suppose that system (1.2), where $g(x)$ and $f(x)$ are arbitrary polynomial, has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form

$$\dot{x} = y, \quad \dot{y} = -x\,(1 + a_1\,x + \ldots + a_{2l}\,x^{2l}) + y\,(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \tag{1.3}$$

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Note that for $g(x) \equiv x$, by the change of variables $X = x$ and $Y = y + F(x)$, where $F(x) = \displaystyle\int_0^x f(s)\,\mathrm{d}s$, (1.3) is reduced to an equivalent system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -X \tag{1.4}$$

which can be written in the form

$$\dot{x} = y, \quad \dot{y} = -x + F(y) \tag{1.5}$$

or

$$\dot{x} = y, \quad \dot{y} = -x + \gamma_1\, y + \gamma_2\, y^2 + \gamma_3\, y^3 + \ldots + \gamma_{2k}\, y^{2k} + \gamma_{2k+1}\, y^{2k+1}. \tag{1.6}$$

In [10,11,16,17], we have presented a solution of Smale's thirteenth problem [25] proving that the Liénard system (1.6) with a polynomial of degree $2k+1$ can have at most k limit cycles and we can conclude now that our results [10, 11, 16, 17] agree with the conjecture of [22] on the maximum number of limit cycles for the classical Liénard polynomial system (1.6). It makes the attempts to construct counterexamples to this conjecture undertaken, e. g., in [4, 5] futile, especially, if these "counterexamples" are absolutely wrong. In [18–20], under some assumptions on the parameters of (1.3), we have found the maximum number of limit cycles and their possible distribution for the general Liénard polynomial system. In [6, 8, 9, 12], we have also presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3:1)$ is their only possible distribution. We have established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems as well. In [7], e. g., we have constructed a canonical cubic dynamical system of Kukles type and have carried out the global qualitative analysis of its special case corresponding to a generalized cubic Liénard equation. In particular, it has been shown that the foci of such a Liénard system can be at most of second order and that such system can have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system have been studied. As a result, a classification of all possible types of separatrix cycles for the generalized cubic Liénard system has been obtained and all possible distributions of its limit cycles have been found. In [13, 14], we have completed the global qualitative analysis of a planar Liénard-type dynamical system with a piecewise linear function containing an arbitrary number of dropping sections and approximating an arbitrary polynomial function. In [2], we have carried out the global qualitative analysis of centrally symmetric cubic systems which are used as learning models of planar neural networks. In [3], we have completed the global qualitative analysis of a quartic dynamical system which models the dynamics of the populations of predators and their prey in a given ecological system. In [15], we have studied multiple limit cycle bifurcations in the well-known FitzHugh–Nagumo neuronal model.

We use the obtained results and develop our methods for studying limit cycle bifurcations of polynomial dynamical systems in this paper as well. In Section 2, applying canonical systems with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we complete the solution of the problem on the maximum number and distribution of limit cycles in the general Liénard polynomial system. This is related to the solution of Hilbert's sixteenth problem on the maximum number and distribution of limit cycles in planar polynomial dynamical systems.

2 Limit Cycle Bifurcations

By means of our bifurcational geometric approach [7, 10, 11, 13, 14, 16–20], we will consider now the general Liénard polynomial system (1.3). The study of singular points of system (1.3) will use two index theorems by H. Poincaré; see [1]. The definition of the Poincaré index is the following [1].

Definition 2.1. Let S be a simple closed curve in the phase plane not passing through a singular point of the system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{2.1}$$

where $P(x, y)$ and $Q(x, y)$ are continuous functions (for example, polynomials), and M be some point on S. If the point M goes around the curve S in the positive direction (counterclockwise) one time, then the vector coinciding with the direction of a tangent to the trajectory passing through the point M is rotated through the angle $2\pi j$ ($j = 0, \pm 1, \pm 2, \ldots$). The integer j is called the *Poincaré index* of the closed curve S relative to the vector field of system (2.1) and has the expression

$$j = \frac{1}{2\pi} \oint_S \frac{P\,dQ - Q\,dP}{P^2 + Q^2}\,. \tag{2.2}$$

According to this definition, the index of a node or a focus, or a center is equal to $+1$ and the index of a saddle is -1. The following Poincaré index theorems are valid [1].

Theorem 2.2. *If N, N_f, N_c, and C are respectively the number of nodes, foci, centers, and saddles in a finite part of the phase plane and N' and C' are the number of nodes and saddles at infinity, then it is valid the formula*

$$N + N_f + N_c + N' = C + C' + 1. \tag{2.3}$$

Theorem 2.3. *If all singular points are simple, then along an isocline without multiple points lying in a Poincaré hemisphere which is obtained by a stereographic projection of the phase plane, the singular points are distributed so that a saddle is followed by a node or a focus, or a center and vice versa. If two points are separated by the equator of the Poincaré sphere, then a saddle will be followed by a saddle again and a node or a focus, or a center will be followed by a node or a focus, or a center.*

Consider system (1.3) supposing that $a_1^2+\ldots+a_{2l}^2\neq 0$. Its finite singularities are determined by the algebraic system

$$x\,(1+a_1\,x+\ldots+a_{2l}\,x^{2l})=0,\quad y=0. \tag{2.4}$$

This system always has an anti-saddle at the origin and, in general, can have at most $2l+1$ finite singularities which lie on the x-axis and are distributed so that a saddle (or saddle-node) is followed by a node or a focus, or a center and vice versa [1]. For studying the infinite singularities, the methods applied in [1] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [6] can be used; see [10, 11, 16–20].

Following [6], we will study limit cycle bifurcations of (1.3) by means of canonical systems containing field rotation parameters of (1.3) [1, 6].

Theorem 2.4. *The Liénard polynomial system* (1.3) *with limit cycles can be reduced to one of the canonical forms:*

$$\begin{gathered}\dot x=y,\\ \dot y=-x\,(1+a_1x+\ldots+a_{2l}x^{2l})\\ +\,y(\alpha_0-\beta_1-\ldots-\beta_{2k-1}+\beta_1x+\alpha_2x^2+\ldots+\beta_{2k-1}x^{2k-1}+\alpha_{2k}x^{2k})\end{gathered} \tag{2.5}$$

or

$$\begin{gathered}\dot x=y\equiv P(x,y),\\ \dot y=x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})\\ +\,y(\alpha_0-\beta_1-\ldots-\beta_{2k-1}+\beta_1x+\alpha_2x^2+\ldots+\beta_{2k-1}x^{2k-1}+\alpha_{2k}x^{2k})\equiv Q(x,y),\end{gathered} \tag{2.6}$$

where $1+a_1x+\ldots+a_{2l}x^{2l}\neq 0$, $\alpha_0,\,\alpha_2,\ldots,\,\alpha_{2k}$ *are field rotation parameters and* $\beta_1,\,\beta_3,\ldots,\,\beta_{2k-1}$ *are semi-rotation parameters.*

Proof. Let us compare system (1.3) with (2.5) and (2.6). It is easy to see that system (2.5) has the only finite singular point: an anti-saddle at the origin. System (2.6) has at list two singular points including an anti-saddle at the origin and a saddle which, without loss of generality, can be always putted into the point $(1,0)$. Instead of the odd parameters $\alpha_1,\,\alpha_3,\ldots,\,\alpha_{2k-1}$ in system (1.3), also without loss of generality, we have introduced new parameters $\beta_1,\,\beta_3,\ldots,\,\beta_{2k-1}$ into (2.5) and (2.6).

We will study now system (2.6) (system (2.5) can be studied absolutely similarly). Let all of the parameters $\alpha_0,\,\alpha_2,\ldots,\,\alpha_{2k}$ and $\beta_1,\,\beta_3,\ldots,\,\beta_{2k-1}$ vanish in this system,

$$\dot x=y,\quad \dot y=x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1}), \tag{2.7}$$

and consider the corresponding equation

$$\frac{dy}{dx}=\frac{x(x-1)(1+b_1x+\ldots+b_{2l-1}x^{2l-1})}{y}\equiv F(x,y). \tag{2.8}$$

Since $F(x, -y) = -F(x, y)$, the direction field of (2.8) (and the vector field of (2.7) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters $b_1, \ldots, b_{2l-1}$ system (2.7) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, we can fix the parameters $b_1, \ldots, b_{2l-1}$ in system (2.6), fixing the position of its finite singularities on the x-axis.

To prove that the even parameters α_0, $\alpha_2, \ldots,$ α_{2k} rotate the vector field of (2.6), let us calculate the following determinants:

$$\Delta_{\alpha_0} = P\,Q'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0,$$
$$\Delta_{\alpha_2} = P\,Q'_{\alpha_2} - QP'_{\alpha_2} = x^2y^2 \geq 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{\alpha_{2k}} = P\,Q'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k}y^2 \geq 0.$$

By definition of a field rotation parameter [1, 6], for increasing each of the parameters α_0, $\alpha_2, \ldots,$ α_{2k}, under the fixed others, the vector field of system (2.6) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.6) is rotated in the negative direction (clockwise).

Calculating the corresponding determinants for the parameters β_1, $\beta_3, \ldots,$ β_{2k-1}, we can see that

$$\Delta_{\beta_1} = P\,Q'_{\beta_1} - QP'_{\beta_1} = (x-1)\,y^2,$$
$$\Delta_{\beta_3} = P\,Q'_{\beta_3} - QP'_{\beta_3} = (x^3-1)\,y^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{\beta_{2k-1}} = P\,Q'_{\beta_{2k-1}} - QP'_{\beta_{2k-1}} = (x^{2k-1}-1)\,y^2.$$

It follows [1, 6] that, for increasing each of the parameters β_1, $\beta_3, \ldots,$ β_{2k-1}, under the fixed others, the vector field of system (2.6) is rotated in the positive direction (counterclockwise) in the half-plane $x > 1$ and in the negative direction (clockwise) in the half-plane $x < 1$ and vice versa for decreasing each of these parameters. We will call these parameters as semi-rotation ones.

Thus, for studying limit cycle bifurcations of (1.3), it is sufficient to consider the canonical systems (2.5) and (2.6) containing the field rotation parameters α_0, $\alpha_2, \ldots,$ α_{2k} and the semi-rotation parameters β_1, $\beta_3, \ldots,$ β_{2k-1}. The theorem is proved. $\square$

By means of the canonical systems (2.5) and (2.6), we will prove the following theorem.

Theorem 2.5. *The Liénard polynomial system* (1.3) *can have at most* $k + l + 1$ *limit cycles,* $k+1$ *surrounding the origin and* l *surrounding one by one the other singularities of* (1.3).

Proof. According to Theorem 2.4, for the study of limit cycle bifurcations of system (1.3), it is sufficient to consider the canonical systems (2.5) and (2.6) containing the field rotation parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ and the semi-rotation parameters $\beta_1, \beta_3, \ldots, \beta_{2k-1}$. We will work with (2.6) again (system (2.5) can be considered in a similar way).

Vanishing all of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ and $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ in (2.6), we will have system (2.7) which is symmetric with respect to the x-axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles or saddle-nodes of (2.7) lying on the x-axis.

Let us input successively the semi-rotation parameters $\beta_1, \beta_3, \ldots, \beta_{2k-1}$ into system (2.7) beginning with the parameters at the highest degrees of x and alternating with their signs. So, begin with the parameter β_{2k-1} and let, for definiteness, $\beta_{2k-1} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(-\beta_{2k-1} + \beta_{2k-1} x^{2k-1}). \end{aligned} \tag{2.9}$$

In this case, the vector field of (2.9) is rotated in the negative direction (clockwise) in the half-plane $x < 1$ turning the center at the origin into a rough stable focus. All of the other centers lying in the half-plane $x > 1$ become rough unstable foci, since the vector field of (2.9) is rotated in the positive direction (counterclockwise) in this half-plane [1, 6].

Fix β_{2k-1} and input the parameter $\beta_{2k-3} < 0$ into (2.9):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(-\beta_{2k-3} - \beta_{2k-1} + \beta_{2k-3} x^{2k-3} + \beta_{2k-1} x^{2k-1}). \end{aligned} \tag{2.10}$$

Then the vector field of (2.10) is rotated in the opposite directions in each of the half-planes $x < 1$ and $x > 1$. Under decreasing β_{2k-3}, when $\beta_{2k-3} = -\beta_{2k-1}$, the focus at the origin becomes nonrough (weak), changes the character of its stability and generates a stable limit cycle. All of the other foci in the half-plane $x > 1$ will also generate unstable limit cycles for some values of β_{2k-3} after changing the character of their stability. Under further decreasing β_{2k-3}, all of the limit cycles will expand disappearing on separatrix cycles of (2.10) [1, 6].

Denote the limit cycle surrounding the origin by Γ_0, the domain outside the cycle by D_{01}, the domain inside the cycle by D_{02} and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter β_{2k-3}, a semi-stable limit cycle cannot appear in the domain D_{02}, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [10, 11, 16–20].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_{01}. Suppose it appears in this domain for some values of the parameters

$\beta^*_{2k-1} > 0$ and $\beta^*_{2k-3} < 0$. Return to system (2.7) and change the inputting order for the semi-rotation parameters. Input first the parameter $\beta_{2k-3} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_{2k-3} + \beta_{2k-3}x^{2k-3}). \end{aligned} \tag{2.11}$$

Fix it under $\beta_{2k-3} = \beta^*_{2k-3}$. The vector field of (2.11) is rotated counterclockwise and the origin turns into a rough unstable focus. Inputting the parameter $\beta_{2k-1} > 0$ into (2.11), we get again system (2.10) the vector field of which is rotated clockwise. Under this rotation, a stable limit cycle Γ_0 will appear from a separatrix cycle for some value of β_{2k-1}. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing β_{2k-1} to the value β^*_{2k-1}. It follows that there are no values of $\beta^*_{2k-3} < 0$ and $\beta^*_{2k-1} > 0$ for which a semi-stable limit cycle could appear in the domain D_{01}.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.10) for any values of the parameters β_{2k-3} and β_{2k-1} of different signs. Obviously, if these parameters have the same sign, system (2.10) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than l limit cycles surrounding the other singularities (foci or nodes) of (2.10) one by one.

It is clear that inputting the other semi-rotation parameters $\beta_{2k-5}, \ldots, \beta_1$ into system (2.10) will not give us more limit cycles, since all of these parameters are rough with respect to the origin and the other anti-saddles lying in the half-plane $x > 1$. Therefore, the maximum number of limit cycles for the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(-\beta_1 - \ldots - \beta_{2k-3} - \beta_{2k-1} + \beta_1 x + \ldots + \beta_{2k-3}x^{2k-3} + \beta_{2k-1}x^{2k-1}) \end{aligned} \tag{2.12}$$

is equal to $l + 1$ and they surround the anti-saddles (foci or nodes) of (2.12) one by one.

Suppose that $\beta_1 + \ldots + \beta_{2k-3} + \beta_{2k-1} > 0$ and input the last rough parameter $\alpha_0 > 0$ into system (2.12):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(\alpha_0 - \beta_1 - \ldots - \beta_{2k-1} + \beta_1 x + \ldots + \beta_{2k-1}x^{2k-1}). \end{aligned} \tag{2.13}$$

This parameter rotating the vector field of (2.13) counterclockwise in the whole phase plane also will not give us more limit cycles, but under increasing α_0, when $\alpha_0 = \beta_1 + \ldots + \beta_{2k-1}$, we can make the focus at the origin nonrough (weak), after the disappearance of the limit cycle Γ_0 in it. Fix this value of the parameter α_0 ($\alpha_0 = \alpha^*_0$):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1}). \end{aligned} \tag{2.14}$$

Let us input now successively the other field rotation parameters $\alpha_2, \ldots, \alpha_{2k}$ into system (2.14) beginning again with the parameters at the highest degrees of x and alternating with their signs; see [10, 11, 16–20]. So, begin with the parameter α_{2k} and let $\alpha_{2k} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.15}$$

In this case, the vector field of (2.15) is rotated clockwise in the whole phase plane and the focus at the origin changes the character of its stability generating again a stable limit cycle. The limit cycles surrounding the other singularities of (2.15) can also still exist. Denote the limit cycle surrounding the origin by Γ_1, the domain outside the cycle by D_1 and the domain inside the cycle by D_2. The uniqueness of a limit cycle surrounding the origin (and limit cycles surrounding the other singularities) for system (2.15) can be proved by contradiction like we have done above for (2.10); see also [10, 11, 16–20].

Let system (2.15) have the unique limit cycle Γ_1 surrounding the origin and l limit cycles surrounding the other antisaddles of (2.15). Fix the parameter $\alpha_{2k} < 0$ and input the parameter $\alpha_{2k-2} > 0$ into (2.15):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-2} x^{2k-2} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.16}$$

Then the vector field of (2.16) is rotated in the opposite direction (counterclockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x changes) generating the second (unstable) limit cycle Γ_2. The limit cycles surrounding the other singularities of (2.16) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter α_{2k-2}. Under further increasing α_{2k-2}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter [10, 11, 16–20].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i.e., that for some values of these parameters, $\alpha^*_{2k} < 0$ and $\alpha^*_{2k-2} > 0$, such a semi-stable cycle exists. Return to system (2.14) again and input first the parameter $\alpha_{2k-2} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-2} x^{2k-2}). \end{aligned} \tag{2.17}$$

This parameter rotates the vector field of (2.17) counterclockwise preserving the origin as a nonrough stable focus.

Fix this parameter under $\alpha_{2k-2} = \alpha^*_{2k-2}$ and input the parameter $\alpha_{2k} < 0$ into (2.17) getting again system (2.16). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k} > \alpha^*_{2k}$, there exists some value of the parameter, α^{12}_{2k} ($\alpha^{12}_{2k} < \alpha^*_{2k} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (2.16) and then splits into a stable cycle Γ_1 and an unstable cycle Γ_2 under further decreasing α_{2k}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k}. The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k} < \alpha^{12}_{2k}$ [10, 11, 16–20].

Thus, there are no such values of the parameters, $\alpha^*_{2k} < 0$ and $\alpha^*_{2k-2} > 0$, for which system (2.16) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters α_{2k} and α_{2k-2} for which system (2.16) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (2.16). Therefore, $l + 2$ is the maximum number of limit cycles in system (2.16).

Suppose that system (2.16) has two limit cycles, Γ_1 and Γ_2, surrounding the origin and l limit cycles surrounding the other antisaddles of (2.16) (this is always possible if $-\alpha_{2k} \gg \alpha_{2k-2} > 0$). Fix the parameters α_{2k}, α_{2k-2} and consider a more general system inputting the third parameter, $\alpha_{2k-4} < 0$, into (2.16):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1 x + \ldots + \beta_{2k-1}x^{2k-1} + \alpha_{2k-4}x^{2k-4} + \alpha_{2k-2}x^{2k-2} + \alpha_{2k}x^{2k}). \end{aligned} \tag{2.18}$$

For decreasing α_{2k-4}, the vector field of (2.18) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3. With further decreasing α_{2k-4}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a “trajectory concentration” surrounding the origin; the cycle Γ_1 will expand disappearing on a separatrix cycle of (2.18).

Let system (2.18) have three limit cycles surrounding the origin: $\Gamma_1, \Gamma_2, \Gamma_3$. Could an additional semi-stable limit cycle appear with decreasing α_{2k-4} after splitting of which system (2.18) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-4}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} < 0$, $\alpha^*_{2k-2} > 0$ and $\alpha^*_{2k-4} < 0$ such a semi-stable cycle exists. Return to system (2.14) again inputting first the parameters $\alpha_{2k-2} > 0$ and $\alpha_{2k-4} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_{2k-4} x^{2k-4} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.19}$$

Fix the parameter α_{2k-2} under the value α^*_{2k-2}. With decreasing α_{2k-4}, a separatrix cycle formed around the origin will generate a stable limit cycle Γ_1. Fix α_{2k-4} under the value α^*_{2k-4} and input the parameter $\alpha_{2k} > 0$ into (2.19) getting system (2.18).

Since, by our assumption, (2.18) has three limit cycles for $\alpha_{2k} > \alpha^*_{2k}$, there exists some value of the parameter α^{23}_{2k} $(\alpha^{23}_{2k} < \alpha^*_{2k} < 0)$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle Γ_2 and a stable cycle Γ_3 with further decreasing α_{2k}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [10, 11, 16–20].

All other combinations of the parameters α_{2k}, α_{2k-2}, and α_{2k-4} are considered in a similar way. It follows that system (2.18) can have at most $l + 3$ limit cycles.

If we continue the procedure of successive inputting the field rotation parameters, $\alpha_{2k}, \ldots, \alpha_2$, into system (2.14),

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1 x + \ldots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\beta_1 x + \ldots + \beta_{2k-1} x^{2k-1} + \alpha_2 x^2 + \ldots + \alpha_{2k} x^{2k}), \end{aligned} \tag{2.20}$$

it is possible to obtain k limit cycles surrounding the origin and l surrounding one by one the other singularities (foci or nodes) $(-\alpha_{2k} \gg \alpha_{2k-2} \gg -\alpha_{2k-4} \gg \alpha_{2k-6} \gg \ldots)$.

Then, by means of the parameter $\alpha_0 \neq \beta_1 + \ldots + \beta_{2k-1}$ $(\alpha_0 > \alpha^*_0$, if $\alpha_2 < 0$, and $\alpha_0 < \alpha^*_0$, if $\alpha_2 > 0)$, we will have the canonical system (2.6) with an additional limit cycle surrounding the origin and can conclude that this system (i.e., the Liénard polynomial system (1.3) as well) has at most $k+l+1$ limit cycles, $k+1$ surrounding the origin and l surrounding one by one the antisaddles (foci or nodes) of (2.6) (and (1.3) as well). The theorem is proved. □

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