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Existence of Solutions for Complementary Lidstone Boundary Value Problems on Time Scales

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Abstract

In this paper, we consider a complementary Lidstone boundary value problem on time scales. Existence of one and two solutions are established using fixed point methods. Examples are given to illustrate our results.

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1 Introduction

Let $\mathbb{T}$ be an arbitrary time scale (nonempty closed subset of $\mathbb{R}$). As usual, $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

Also $x^\sigma(t) = x(\sigma(t))$, and $x^\Delta(t)$ denotes the time scale derivative of x. Higher order jump and derivative are defined inductively by $\sigma^j(t) = \sigma(\sigma^{j-1}(t))$ and $x^{\Delta^{(j)}}(t) = (x^{\Delta^{(j-1)}}(t))^\Delta$, $j \geq 1$. It is assumed that the reader is familiar with the time scale calculus. Some preliminary definitions and theorems on time scales can be found in [1–4].

In this paper, we discuss the existence of solutions to the complementary Lidstone boundary value problem (CLBVP) on time scales

$$\begin{aligned} &(-1)^n x^{\Delta^{(2n+1)}}(t) + q(t)f(t, x^\sigma(t)) = 0, \ \ t \in [a, b]_{\mathbb{T}}, \\ &x(a) = 0, \ \ x^{\Delta^{(2i-1)}}(a) = x^{\Delta^{(2i-1)}}(\sigma^{2n-2i+2}(b)) = 0 \ \ i = 1, \dots, n, \end{aligned} \tag{1.1}$$

where $n \geq 1$, $a, b \in \mathbb{T}$ and $f : [a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous. Hereafter, we use the notation $[a, b]_{\mathbb{T}}$ to indicate the time scale interval $[a, b] \cap \mathbb{T}$. The interval $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ and $(a, b)_{\mathbb{T}}$ are similarly defined.

The complementary Lidstone interpolation and boundary value problems were discussed in [5–7]. In [6, 7] the authors consider a $(2n + 1)$th order differential equation together with boundary data at the odd order derivatives

$$x(0) = a_0, \ \ x^{(2i-1)}(0) = a_i, \ \ x^{(2i-1)}(1) = b_i, \ \ i = 1, 2, \dots, n. \tag{1.2}$$

The boundary conditions (1.2) are known as complementary Lidstone boundary conditions, which naturally complement the Lidstone boundary conditions [8–11] which involve even order derivatives. The Lidstone boundary value problem comprises a $2m$th order differential equation and the Lidstone boundary conditions

$$x^{(2i)}(0) = a_i, \ \ x^{(2i)}(1) = b_i, \ \ i = 0, 2, \dots, n - 1.$$

For the Lidstone boundary value problem, we refer the reader to [12–26] and the references cited therein. For the complementary Lidstone boundary value problem, we refer the reader to [27–31]. In the literature there are only a few papers on Lidstone boundary value problems on time scales [32, 33] and no paper (to our knowledge) on complementary Lidstone boundary value problems on time scales.

In Section 2, we develop some inequalities for certain Green's functions. In Section 3, using a variety of fixed point theorems, we establish of existence of a solution (not necessary positive), and we also discuss the existence of a nontrivial positive solution, and two nontrivial positive solutions.

2 Preliminaries

To obtain a solution for the CLBVP (1.1), we require a mapping whose kernel $G_n^1(t,s)$ is the Green's function of the Lidstone boundary value problem

$$\begin{aligned}&(-1)^n y^{\Delta^{(2n)}}(t) = 0,\ \ t\in[a,b]_{\mathbb{T}},\\ &y^{\Delta^{(2i)}}(a) = y^{\Delta^{(2i)}}(\sigma^{2n-2i}(b)) = 0,\ \ i = 0,1,\ldots,n-1.\end{aligned} \tag{2.1}$$

The Green's function can be expressed as

$$G_n^1(t,s) = \int_a^{\sigma^{2n-1}(b)} G_n(t,r)G_{n-1}^1(r,s)\Delta r, \tag{2.2}$$

where

$$G_n(t,s) = \frac{-1}{\sigma^{2n}(b)-a}\begin{cases}(t-a)(\sigma^{2n}(b)-\sigma(s)), & t\le s,\\ (\sigma(s)-a)(\sigma^{2n}(b)-t), & \sigma(s)<t,\end{cases} \tag{2.3}$$

and

$$G_1^1(t,s) = G_1(t,s). \tag{2.4}$$

We remark that G_n is the Green's function of the problem

$$y^{\Delta\Delta}(t) = 0,\ y(a) = y(\sigma^{2n}(b)) = 0.$$

Furthermore, it is easily seen that from (2.3), we have

$$G_n(t,s)\le 0,\ \ (t,s)\in[a,\sigma^{2n}(b)]_{\mathbb{T}}\times[a,\sigma^{2n-2}(b)]_{\mathbb{T}}, \tag{2.5}$$

and from (2.5) and (2.2), we have

$$(-1)^n G_n^1(t,s)\ge 0,\ \ (t,s)\in[a,\sigma^{2n}(b)]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}. \tag{2.6}$$

Lemma 2.1. *For $(t,s)\in[a,\sigma^{2n}(b)]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$, we have*

$$(-1)^n G_n^1(t,s) = |G_n^1(t,s)| \le \theta_n(\sigma(s)-a)(\sigma^2(b)-\sigma(s)), \tag{2.7}$$

where

$$\theta_n = \left[\prod_{i=1}^{n}(\sigma^{2i}(b)-a)\right]^{-1}\prod_{i=1}^{n-1} s_{2i},$$

$$s_j \ = \ \frac{1}{6}\Bigg\{(\sigma^{j+2}(b)-a)^3 + \sum_{t\in A_j}\mu(t)^2[3(\sigma^{j+2}(b)+a)-2(t+2\sigma(t))]$$

$$-\sum_{t\in B_j}\mu(t)^2[2(t+2\sigma(t))-3(\sigma^{j+2}(b)+a)]\Big\},\quad j\geq 2 \tag{2.8}$$

and

$$\begin{aligned} A_j &= \Big[a,\frac{\sigma^{j+1}(b)+a}{2}\Big]_{\mathbb{T}}\setminus\Big\{\max\big\{t:t\in\big[a,\frac{\sigma^{j+1}(b)+a}{2}\big]_{\mathbb{T}}\big\}\Big\},\\ B_j &= \Big(\frac{\sigma^{j+1}(b)+a}{2},\sigma^{j+1}(b)\Big]_{\mathbb{T}}\cup\Big\{\max\{t:t\in\big[a,\frac{\sigma^{j+1}(b)+a}{2}\big]_{\mathbb{T}}\}\Big\}. \end{aligned} \tag{2.9}$$

Proof. From (2.3), we obtain for $n\geq 1$ and $(t,s)\in[a,\sigma^{2n}(b)]_{\mathbb{T}}\times[a,\sigma^{2n-2}(b)]_{\mathbb{T}}$ that

$$\begin{aligned} |G_n(t,s)| &\leq \frac{1}{\sigma^{2n}(b)-a}\begin{cases}(s-a)(\sigma^{2n}(b)-\sigma(s)), & t\leq s,\\ (\sigma(s)-a)(\sigma^{2n}(b)-\sigma(s)), & \sigma(s)<t\end{cases}\\ &\leq \frac{1}{\sigma^{2n}(b)-a}(\sigma(s)-a)(\sigma^{2n}(b)-\sigma(s)). \end{aligned} \tag{2.10}$$

In view of (2.4) and (2.10) $|_{n=1}$, we see that (2.7) is true for $n=1$. Assume that (2.7) holds for $n=k(\geq 1)$. Then for $(t,s)\in[a,\sigma^{2k+2}(b)]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$, it follows from (2.2), (2.5), (2.6), (2.10)$|_{n=k+1}$ and (2.7)$|_{n=k}$ that

$$\begin{aligned} |G^1_{k+1}(t,s)| &= \int_a^{\sigma^{2k+1}(b)}|G_{k+1}(t,r)||G^1_k(r,s)|\Delta r\\ &\leq \int_a^{\sigma^{2k+1}(b)}\frac{1}{\sigma^{2k+2}(b)-a}(\sigma(r)-a)(\sigma^{2k+2}(b)-\sigma(r))\theta_k(\sigma(s)-a)(\sigma^2(b)-\sigma(s))\Delta r\\ &= \frac{1}{\sigma^{2k+2}(b)-a}\theta_k(\sigma(s)-a)(\sigma^2(b)-\sigma(s))\int_a^{\sigma^{2k+1}(b)}(\sigma(r)-a)(\sigma^{2k+2}(b)-\sigma(r))\Delta r\\ &= \frac{1}{\sigma^{2k+2}(b)-a}\theta_k(\sigma(s)-a)(\sigma^2(b)-\sigma(s))\Big\{\int_a^{\sigma^{2k+2}(b)}(r-a)(\sigma^{2k+2}(b)-r)dr\\ &\quad+\sum_{t\in A_{2k}}\int_t^{\sigma(t)}[(\sigma(t)-a)(\sigma^{2k+2}(b)-\sigma(t))-(r-a)(\sigma^{2k+2}(b)-r)]dr\\ &\quad-\sum_{t\in B_{2k}}\int_t^{\sigma(t)}[(r-a)(\sigma^{2k+2}(b)-r)-(\sigma(t)-a)(\sigma^{2k+2}(b)-\sigma(t))]dr\Big\}\\ &= \frac{\theta_k(\sigma(s)-a)(\sigma^2(b)-\sigma(s))}{6(\sigma^{2k+2}(b)-a)}\Big\{(\sigma^{2k+2}(b)-a)^3\\ &\quad+\sum_{t\in A_{2k}}\mu(t)^2[3(\sigma^{2k+2}(b)+a)-2(t+2\sigma(t))]\\ &\quad-\sum_{t\in B_{2k}}\mu(t)^2[2(t+2\sigma(t))-3(\sigma^{2k+2}(b)+a)]\Big\} \end{aligned}$$

$$\begin{aligned} &= \frac{\theta_k s_{2k}}{\sigma^{2k+2}(b)-a}(\sigma(s)-a)(\sigma^2(b)-\sigma(s)) \\ &= \theta_{k+1}(\sigma(s)-a)(\sigma^2(b)-\sigma(s)). \end{aligned}$$

Thus the inequality (2.7) is true for $n = k+1$. This concludes the proof. □

Remark 2.2. If $\mathbb{T} = \mathbb{R}$, then from Lemma 2.1, we obtain for $(t,s) \in [a,b] \times [a,b]$

$$(-1)^n G_n^1(t,s) = |G_n^1(t,s)| \le \left(\frac{(b-a)^2}{6}\right)^{n-1} \frac{(s-a)(b-s)}{b-a}. \tag{2.11}$$

Lemma 2.3. *Let $\delta \in \left(0, \frac{1}{2}\right)$ be a given constant. For $(t,s) \in [\alpha, \beta_n]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$, we have*

$$(-1)^n G_n^1(t,s) = |G_n^1(t,s)| \ge \psi_n(\delta)(\sigma(s)-a)(\sigma^2(b)-\sigma(s)), \tag{2.12}$$

where

$$\alpha = \min\left\{t \in [a, \sigma^{2n}(b)]_{\mathbb{T}} : a + \delta \le t\right\},$$
$$\beta_j = \max\left\{t \in [a, \sigma^{2j}(b)]_{\mathbb{T}} : t \le \sigma^{2j}(b) - \delta\right\},$$

$$\psi_n(\delta) = \delta^n \prod_{i=1}^{n} (\sigma^{2i}(b)-a) \prod_{i=1}^{n-1} S_{i+1}$$

and

$$\begin{aligned} S_j = \;& \frac{1}{6}\Big\{(\beta_j - \alpha)(3\sigma^{2j}(b)(\beta_j + \alpha) + 3a(\beta_j + \alpha - 2\sigma^{2j}(b)) - 2(\beta_j^2 + \beta_j\alpha + \alpha^2) \\ &+ \sum_{t \in A_j \setminus [a,\alpha)_{\mathbb{T}}} \mu(t)^2[3(\sigma^{2j}(b)+a) - 2(t + 2\sigma(t))] \\ &- \sum_{t \in B_j \setminus (\beta_j, \sigma^{2j}(b)]_{\mathbb{T}}} \mu(t)^2[2(t+2\sigma(t)) - 3(\sigma^{2j}(b)+a)]\Big\}, \quad j \ge 2. \end{aligned}$$

Here the sets A_j and B_j are defined as in (2.9).

Proof. For $n \ge 1$ and $(t,s) \in [\alpha, \beta_n]_{\mathbb{T}} \times [a, \sigma^{2n-2}(b)]_{\mathbb{T}}$, from (2.3), we have

$$\left|\frac{G_n(t,s)}{G_n(\sigma(s),s)}\right| = \frac{t-a}{\sigma(s)-a} \ge \frac{\alpha - a}{\sigma(s)-a} \ge \frac{a+\delta-a}{\sigma(b)-a} \ge \frac{\delta}{\sigma^{2n}(b)-a}, \quad \text{if } t \le s$$

and

$$\left|\frac{G_n(t,s)}{G_n(\sigma(s),s)}\right| = \frac{\sigma^{2n}(b)-t}{\sigma^{2n}(b)-\sigma(s)} \ge \frac{\sigma^{2n}(b)-\beta_n}{\sigma^{2n}(b)-\sigma(s)a}$$

$$\geq \quad \frac{\sigma^{2n}(b)-\sigma^{2n}(b)+\delta}{\sigma^{2n}(b)-a} = \frac{\delta}{\sigma^{2n}(b)-a}, \quad \text{if } t > \sigma(s).$$

Hence, we obtain

$$|G_n(t,s)| \geq \frac{\delta}{\sigma^{2n}(b)-a}(\sigma(s)-a)(\sigma^{2n}(b)-\sigma(s)). \tag{2.13}$$

Noting (2.4) and (2.13)$|_{n=1}$, we see that (2.12) is true for $n = 1$. Now suppose that (2.12) holds for $n = k(\geq 1)$. Then using (2.2), (2.5), (2.6), (2.13)$|_{n=k+1}$ and (2.12)$|_{n=k}$, we get for $(t,s) \in [\alpha, \beta_n]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$,

$$\begin{aligned}
|\,G^1_{k+1}(t,s)| &= \int_a^{\sigma^{2k+1}(b)} |G_{k+1}(t,r)||G^1_k(r,s)|\Delta r \\
&\geq \int_a^{\sigma^{2k+1}(b)} \frac{\delta}{\sigma^{2k+2}(b)-a}(\sigma(r)-a)(\sigma^{2k+2}(b)-\sigma(r))G^1_k(r,s)\Delta r \\
&\geq \int_\alpha^{\beta_{k+1}} \frac{\delta}{\sigma^{2k+2}(b)-a}(\sigma(r)-a)(\sigma^{2k+2}(b)-\sigma(r))\psi_k(\delta)(\sigma(s)-a)(\sigma^2(b)-\sigma(s))\Delta r \\
&\geq \frac{\delta\psi_k(\delta)}{\sigma^{2k+2}(b)-a}(\sigma(s)-a)(\sigma^2(b)-\sigma(s)) \int_\alpha^{\beta_{k+1}} (\sigma(r)-a)(\sigma^{2k+2}(b)-\sigma(r))\Delta r \\
&\geq \frac{\delta\psi_k(\delta)}{\sigma^{2k+2}(b)-a}(\sigma(s)-a)(\sigma^2(b)-\sigma(s))\Bigg\{ \int_\alpha^{\beta_{k+1}} (r-a)(\sigma^{2k+2}(b)-r)dr \\
&\quad + \sum_{t\in A_{2k}\setminus[a,\alpha)_{\mathbb{T}}} \int_t^{\sigma(t)} [(\sigma(t)-a)(\sigma^{k+1}(b)-\sigma(t))-(r-a)(\sigma^{2k+2}(b)-r)]dr \\
&\quad - \sum_{t\in B_{2k}\setminus(\beta_{k+1},\sigma^{2k+2}(b)]_{\mathbb{T}}} \int_t^{\sigma(t)} [(r-a)(\sigma^{2k+2}(b)-r)-(\sigma(t)-a)(\sigma^{2k+2}(b)-\sigma(t))]dr \Bigg\} \\
&= \frac{\delta\psi_k(\delta)}{\sigma^{2k+2}(b)-a}(\sigma(s)-a)(\sigma^2(b)-\sigma(s))S_{k+1} \\
&= \psi_{k+1}(\delta)(\sigma(s)-a)(\sigma^2(b)-\sigma(s)).
\end{aligned}$$

Hence, (2.12) is true for $n = k+1$. This concludes the proof. □

Remark 2.4. If $\mathbb{T} = \mathbb{R}$, then from Lemma 2.3, we obtain for $(t,s) \in [a+\delta, b-\delta] \times [a,b]$

$$(-1)^n G^1_n(t,s) = |G^1_n(t,s)| \geq \psi_n(\delta)\frac{(s-a)(b-s)}{b-a}, \tag{2.14}$$

where $\psi_n(\delta) = \dfrac{1}{6^{n-1}}\left(\dfrac{\delta}{b-a}\right)^n \left((b-a)^2 - 6\delta^2 + \dfrac{4\delta^3}{b-a}\right)^{n-1}$.

Remark 2.5. From Lemma 2.1 and Lemma 2.3, for $\delta = \dfrac{1}{4} \in \left(0, \dfrac{1}{2}\right)$, we have

$$\begin{aligned}\min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} |G_n^1(t,s)| \geq \psi_n(1/4)G_1(\sigma(s),s) &\geq \frac{\psi_n(\frac{1}{4})}{\theta_n} \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} |G_n^1(t,s)| \\ &\geq \gamma_n \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} |G_n^1(t,s)|,\end{aligned}$$

where

$$\gamma_n = \frac{\prod_{i=1}^{n-1} S_{i+1}}{4^n \prod_{i=1}^{n-1} s_{2i}}.$$

It is clear that $s_{2i} > S_{i+1}$, $1 \leq i \leq n-1$. Thus, we have $0 < \gamma_n < 1$.

Finally in this section, we state Krasnosel'skii's fixed point theorem in a cone [34, 35].

Theorem 2.6. *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $P \subset B$ be a cone in B. Suppose that Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a continuous and compact operator such that either*

i $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*

ii $\|Tu\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Existence of Positive Solutions

To discuss (1.1), we first consider the initial value problem

$$x^{\Delta}(t) = y(t), \;\; t \in [a, \sigma^{2n}(b)]_{\mathbb{T}} \tag{3.1}$$

$$x(a) = 0 \tag{3.2}$$

whose solution is

$$x(t) = \int_a^t y(s)\Delta s, \;\; t \in [a, \sigma^{2n+1}(b)]_{\mathbb{T}}. \tag{3.3}$$

Taking into account (3.1) and (3.3), the complementary Lidstone boundary value problem (1.1) reduces to the Lidstone boundary value problem

$$\begin{aligned}&(-1)^n y^{\Delta^{(2n)}}(t) + q(t)f\left(t, \int_a^{\sigma(t)} y(s)\Delta s\right) = 0, \;\; t \in [a,b]_{\mathbb{T}}, \\ &y^{\Delta^{(2i)}}(a) = y^{\Delta^{(2i)}}(\sigma^{2n-2i}(b)) = 0, \;\; i = 0, 1, \ldots, n-1.\end{aligned} \tag{3.4}$$

If (3.4) has a solution y^*, then from (3.3),

$$x^*(t) = \int_a^t y^*(s)\Delta s \tag{3.5}$$

is a solution of (1.1). Hence, the existence of a solution of the complementary Lidstone boundary value problem (1.1) follows from the existence of a solution of the Lidstone boundary value problem (3.4). It is clear from (3.5) that

$$\max_{t\in[a,\sigma^{2n+1}(b)]_{\mathbb{T}}} |x^*(t)| \leq (\sigma^{2n}(b)-a) \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} |y^*(t)|,$$

and moreover if y^* is positive, so also is x^*.

Let the Banach space $B = \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}}$ be equipped with the norm

$$\|y\| = \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} |y(t)|$$

for $y \in B$. We now define a mapping $T : \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}} \to \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}}$ by

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s)q(s)f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &= \int_a^{\sigma(b)} |G_n^1(t,s)|q(s)f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s, \end{aligned} \tag{3.6}$$

where $G_n^1(t,s)$ is the Green's function given in (2.2). A fixed point y^* of the operator T is clearly a solution of the boundary value problem (3.4), so $x^*(t) = \int_a^t y^*(s)\Delta s$ is a solution of (1.1). Let

$$K = \{y \in B : y(t) \geq 0, t \in [a,\sigma^{2n}(b)]_{\mathbb{T}}\}.$$

We now list some conditions which will be used in some results in this paper:

(C_1) $q \in \mathcal{C}(([a,\sigma(b)]_{\mathbb{T}}),[0,\infty))$ is not identically zero on any subinterval of $[a,\sigma(b)]_{\mathbb{T}}$, and

$$0 < \int_\nu^\xi G_1(\sigma(s),s)q(s)\Delta s < \infty,$$

where $\nu = \max\left\{t \in [\alpha,\xi]_{\mathbb{T}} : t \leq \dfrac{\alpha+\xi}{2}\right\}$ and $\xi = \min\{\sigma(b),\beta_n\}$.

(C_2) f is continuous on $[a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R}$ and nondecreasing in the second argument with $f(t,u) \geq 0$ for $(t,u) \in [a,\sigma(b)]_{\mathbb{T}} \times K$.

Our first result is an existence criterion for a solution (need not be positive).

Theorem 3.1. *Let* (C_1) *hold and let* f *be continuous. If* $M > 0$ *satisfies* $\theta_n \Lambda Q s_0 \le M$*, where* $Q > 0$ *satisfies*

$$Q \ge \max_{\|y\| \le M} \left| f(t, \int_a^{\sigma(t)} y(s)\Delta s) \right|, \quad for \;\; t \in [a, \sigma(b)]_{\mathbb{T}},$$

$$\Lambda = \max_{t \in [a,\sigma(b)]_{\mathbb{T}}} q(t), \tag{3.7}$$

and the number s_0 *is defined in* $(2.8)|_{j=0}$*, then CLBVP* (1.1) *has a solution* $x^* \in \mathcal{C}[a, \sigma^{2n+1}(b)]_{\mathbb{T}}$ *such that* $\max_{t \in [a,\sigma^{2n+1}(b)]_{\mathbb{T}}} |x^*(t)| \le (\sigma^{2n}(b) - a)M.$

Proof. Let $K_1 = \{y \in B : \|y\| \le M\}$. We will apply Schauder's fixed point theorem. The solutions of problem (3.4) are the fixed points of the operator T. A standard argument guarantees that $T : K_1 \to B$ is continuous. Next we show $T(K_1) \subset K_1$. For $y \in K_1$, we obtain

$$\begin{aligned}
|Ty(t)| &= \left| \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right) \Delta s \right| \\
&\le \int_a^{\sigma(b)} |G_n^1(t,s)| q(s) \left| f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right) \right| \Delta s \\
&\le \theta_n \Lambda Q \int_a^{\sigma(b)} (\sigma(s) - a)(\sigma^2(b) - \sigma(s))\Delta s \\
&\le \theta_n \Lambda Q s_0 \\
&\le M
\end{aligned}$$

for all $t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}$. This implies that $\|Ty\| \le M$. A standard argument, via the Arzela–Ascoli theorem, guarantees that $T : K_1 \to K_1$ is a compact operator. Hence T has a fixed point $y^* \in K_1$ by Schauder's fixed point theorem, and note y^* is a solution of (3.4). From (3.5), it is easy to see that (1.1) has a solution $x^*(t) = \int_a^t y^*(s)\Delta s$ with $\frac{1}{(\sigma^{2n}(b) - a)} \max_{t \in [a,\sigma^{2n+1}(b)]_{\mathbb{T}}} |x^*(t)| \le \|y^*\| \le M.$ □

Corollary 3.2. *If* f *is continuous and bounded on* $[a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R}$*, then the CLBVP* (1.1) *has a solution.*

Next let

$$P = \left\{ y \in B : \min_{t \in [a,\sigma^{2n}(b)]_{\mathbb{T}}} y(t) \ge 0 \;\text{ and }\; \min_{t \in [\alpha,\beta_n]_{\mathbb{T}}} y(t) \ge \gamma_n \|y\| \right\} \subset K. \tag{3.8}$$

It is easy to check that P is a cone of nonnegative functions in $\mathcal{C}[a, \sigma^{2n}(b)]_{\mathbb{T}}$. Now assume ($C_1$) and ($C_2$) hold. Next we will apply Theorem 2.6. First we show $T : P \to P$

(see (3.6) for the definition of T). Now (C_1) and (C_2), $y \in P$ implies that $Ty(t) \geq 0$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$ and

$$\begin{aligned} \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} Ty(t) &= \int_a^{\sigma(b)} \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} (-1)^n G_n^1(t,s)q(s)f\Big(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\Big)\Delta s \\ &\geq \int_a^{\sigma(b)} \gamma_n \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} |G_n(t,s)|q(s)f\Big(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\Big)\Delta s. \end{aligned}$$

It follows that

$$\min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} Ty(t) \geq \gamma_n \|Ty\|.$$

Thus $Ty \in P$ which means $T(P) \subset P$. A standard argument, via the Arzela–Ascoli theorem, guarantees that $T : P \to P$ is continuous and completely continuous.

Theorem 3.3. *Let* (C_1) *and* (C_2) *hold. Also assume*

(C_3) $\lim\limits_{y\to 0^+} \dfrac{f(t,y)}{y} = 0$, $\lim\limits_{y\to\infty} \dfrac{f(t,y)}{y} = \infty$, *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$.

Then the CLBVP (1.1) *has at least one positive solution.*

Proof. We will apply Theorem 2.6 with the cone P defined in (3.8). Since

$$\lim_{y\to 0^+} \frac{f(t,y)}{y} = 0,$$

there exists an $r_1 > 0$ such that

$$f(t,y) \leq \eta y, \ \ 0 \leq y \leq r_1, \ \ a \leq t \leq \sigma(b), \tag{3.9}$$

where $\eta = \dfrac{1}{\theta_n \Lambda(\sigma(b)-a)s_0}$ and the numbers Λ and s_0 are defined in (3.7) and $(2.8)|_{j=0}$, respectively. Let $\Omega_1 = \left\{ y \in B : \|y\| < \dfrac{r_1}{\sigma(b)-a} \right\}$. For $y \in P \cap \partial\Omega_1$, we have

$$\int_a^{\sigma(b)} y(\tau)\Delta\tau \leq \int_a^{\sigma(b)} \|y\|\Delta\tau \leq \frac{r_1}{\sigma(b)-a}(\sigma(b)-a) = r_1. \tag{3.10}$$

Using Lemma 2.1, (3.10) and (3.9), we find for $t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}$ that

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s)q(s)f\Big(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\Big)\Delta s \\ &\leq \eta \int_a^{\sigma(b)} \theta_n G_1(\sigma(s),s)q(s)\Big(\int_a^{\sigma(b)} y(\tau)\Delta\tau\Big)\Delta s \end{aligned}$$

$$\begin{aligned}
&\leq \eta \int_a^{\sigma(b)} \theta_n G_1(\sigma(s), s) q(s) r_1 \Delta s \\
&\leq \Lambda \eta \theta_n r_1 \int_a^{\sigma(b)} G_1(\sigma(s), s) \Delta s \\
&\leq \theta_n \Lambda \eta r_1 s_0 = \frac{r_1}{(\sigma(b) - a)} = \|y\|,
\end{aligned}$$

and so

$$\|Ty\| \leq \|y\| \text{ for all } y \in P \cap \partial\Omega_1. \tag{3.11}$$

Since $\lim\limits_{y\to\infty} \dfrac{f(t,y)}{y} = \infty$, there exists an $\overline{R} > 0$ such that

$$f(t,y) \geq \mu y, \ \ y \geq \overline{R}, \ \ a \leq t \leq \sigma(b), \tag{3.12}$$

where $\mu = \left((\nu - \alpha)\gamma_n \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s), s) q(s) \Delta s \right)^{-1}$. Let

$$R_1 = \max\left\{ \frac{2r_1}{\sigma(b) - a}, \frac{\overline{R}}{(\nu - \alpha)\gamma_n} \right\}$$

and $\Omega_2 = \{y \in B : \|y\| < R_1\}$. For $y \in P \cap \partial\Omega_2$, we have

$$\int_\alpha^\nu y(\tau)\Delta\tau \geq \int_\alpha^\nu \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} y(\tau)\Delta\tau \geq (\nu - \alpha)\gamma_n \|y\| = (\nu - \alpha)\gamma_n R_1 \geq \overline{R}. \tag{3.13}$$

Using Lemma 2.3, (C_2), (3.12) and (3.13) we find for $t_0 \in [\alpha, \beta_n]_{\mathbb{T}}$ that

$$\begin{aligned}
Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0, s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau \right) \Delta s \\
&\geq \int_\alpha^\xi \psi_n(1/4) G_1(\sigma(s), s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau \right) \Delta s \\
&\geq \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s), s) q(s) f\left(s, \int_\alpha^\nu y(\tau)\Delta\tau \right) \Delta s \\
&\geq \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s), s) q(s) \mu \left(\int_\alpha^\nu y(\tau)\Delta\tau \right) \Delta s \\
&\geq \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s), s) q(s) \mu (\nu - \alpha)\gamma_n R_1 \Delta s \\
&\geq \psi_n(1/4) \mu (\nu - \alpha)\gamma_n R_1 \int_\nu^\xi G_1(\sigma(s), s) q(s) \Delta s \\
&\geq R_1 = \|y\|,
\end{aligned}$$

and so

$$\|Ty\| \geq \|y\| \text{ for all } y \in P \cap \partial\Omega_2. \tag{3.14}$$

Consequently, Theorem 2.6 guarantees that T has a fixed point $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. From (3.3), note a positive solution of (1.1) is $x(t) = \int_a^t y(s)\Delta s$. □

Theorem 3.4. *Let* (C_1) *and* (C_2) *hold. Also assume*

(C_4) $\lim_{y\to 0^+} \frac{f(t,y)}{y} = \infty$, $\lim_{y\to\infty} \frac{f(t,y)}{y} = 0$, *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$.

Then the CLBVP (1.1) *has at least one positive solution.*

Proof. Since $\lim_{y\to 0^+} \frac{f(t,y)}{y} = \infty$, there exists an $r_2 > 0$ such that

$$f(t,y) \geq \overline{\mu} y, \quad 0 < y \leq r_2, \quad a \leq t \leq \sigma(b),$$

where $\overline{\mu} \geq \mu$; here μ is given in the proof of Theorem 3.3. Let

$$\Omega_1 = \left\{ y \in B : \|y\| < \frac{r_2}{\nu - \alpha} \right\}.$$

For $y \in P \cap \partial\Omega_1$, we have for $t_0 \in [\alpha, \beta_n]_{\mathbb{T}}$ that

$$\begin{aligned}
Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0,s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\
&\geq \int_\alpha^\xi \psi_n(1/4) G_1(\sigma(s),s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\
&\geq \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s),s) q(s) f\left(s, \int_\alpha^\nu y(\tau)\Delta\tau\right)\Delta s \\
&\geq \psi_n(1/4) \int_\nu^\xi G_1(\sigma(s),s) q(s) \overline{\mu} \left(\int_\alpha^\nu y(\tau)\Delta\tau\right)\Delta s \\
&\geq \psi_n(1/4) \gamma_n r_2 \overline{\mu} \int_\nu^\xi G_1(\sigma(s),s) q(s) \Delta s \\
&\geq \frac{r_2}{\nu - \alpha} = \|y\|,
\end{aligned}$$

and so $\|Ty\| \geq |y\|$ for all $y \in P \cap \partial\Omega_1$. Since $\lim_{y\to\infty} \frac{f(t,y)}{y} = 0$, there exists an $\overline{r_2}$ such that

$$f(t,y) \leq \overline{\eta} y, \quad y \geq \overline{r_2}, \quad a \leq t \leq \sigma(b), \tag{3.15}$$

where $\overline{\eta} = (\theta_n \Lambda(\sigma(b) - a)s_0)^{-1}$.

Case 1. Suppose that f is bounded. Then, there exists some $N > 0$ such that

$$f(t, y) \leq N, \ \ t \in [a, \sigma(b)], \ y \in [0, \infty). \tag{3.16}$$

Let $r_3 = \max\{r_2 + 1, N\theta_n \Lambda s_0\}$ and $\Omega_2 = \{y \in B : \|y\| < r_3\}$. For $y \in P \cap \partial\Omega_2$, using Lemma 2.1, (C$_2$) and (3.16), we get

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t, s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &\leq N\theta_n \Lambda \int_a^{\sigma(b)} G_1(\sigma(s), s)\Delta s \\ &\leq N\theta_n \Lambda s_0 \leq r_3 = \|y\|. \end{aligned}$$

Hence, $\|Ty\| \leq \|y\|$ for all $y \in P \cap \partial\Omega_2$.

Case 2. Suppose that f is unbounded. In this case let

$$g(r) := \max\{f(t, y) : t \in [a, \sigma(b)], 0 \leq y \leq r\}$$

such that $\lim_{r\to\infty} g(r) = \infty$. We choose $r_3 > \max\left\{\dfrac{2r_2}{\nu - \alpha}, \dfrac{\overline{r_2}}{(\nu - \alpha)\gamma_n}\right\}$ such that $g(r_3) \geq g(r)$ and let $\Omega_2 = \{y \in B : \|y\| < r_3\}$. For $y \in P \cap \partial\Omega_2$, we have using Lemma 2.1, (C$_2$) and (3.15) that

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t, s) q(s) f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &\leq \overline{\eta}\theta_n \int_a^{\sigma(b)} G_1(\sigma(s), s) q(s) \left(\int_a^{\sigma(b)} y(\tau)\Delta\tau\right)\Delta s \\ &\leq \overline{\eta}\theta_n \Lambda \|y\| (\sigma(b) - a) \int_a^{\sigma(b)} G_1(\sigma(s), s)\Delta s \\ &\leq \overline{\eta}\theta_n \Lambda(\sigma(b) - a)s_0\|y\| = \|y\|, \end{aligned}$$

and so $\|Ty\| \leq \|y\|$ for all $y \in P \cap \partial\Omega_2$. It follows from Theorem 2.6 that T has a fixed point $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. From (3.3), note a positive solution of (1.1) is $x(t) = \displaystyle\int_a^t y(s)\Delta s$. □

Theorem 3.5. *Let* (C_1) *and* (C_2) *hold. Also assume*

(C_5) $\displaystyle\lim_{y\to 0^+} \frac{f(t, y)}{y} = \infty, \ \lim_{y\to\infty} \frac{f(t, y)}{y} = \infty,$ *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$.

(C_6) *There exists constant* ρ_1 *such that* $f(t, y) \leq \Gamma\rho_1, \ \ for \ \ y \in [0, (\sigma(b) - a)\rho_1]$, *where* $\Gamma = \dfrac{1}{\theta_n \Lambda s_0}$.

Then the LBVP (3.4) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| \le \rho_1 < \|y_2\|$$

and the CLBVP (1.1) *at least two positive solutions* x_1 *and* x_2 *with* $x_1(t) = \int_a^t y_1(s)\Delta s$ *and* $x_2(t) = \int_a^t y_2(s)\Delta s$.

Proof. Since $\lim_{y\to 0^+} \dfrac{f(t,y)}{y} = \infty$ there exists $\rho_* \in (0, \rho_1)$ such that

$$f(t,y) \ge \mu_1 y \;\text{ for }\; 0 \le y \le \rho_*,\;\; a \le t \le \sigma(b), \tag{3.17}$$

where $\mu_1 \ge \mu$; here μ is given in the proof of Theorem 3.3. Choose

$$\overline{\rho_*} = \min\left\{\frac{\rho_*}{\nu-\alpha}, \rho_1\right\}$$

and set $\Omega_1 = \{y \in B : \|y\| < \overline{\rho_*}\}$. For $y \in P \cap \partial\Omega_1$, we have

$$\int_\alpha^\nu y(\tau)\Delta\tau \le \int_\alpha^\nu \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} y(t)\Delta\tau \le (\nu-\alpha)\overline{\rho_*} \le \rho_*. \tag{3.18}$$

Using Lemma 2.3, (C_2), (3.17) and (3.18), we find for $t_0 \in [\alpha, \beta_n]_{\mathbb{T}}$ that

$$\begin{aligned}
Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0,s)q(s)f\Big(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\Big)\Delta s \\
&\ge \int_\alpha^\xi \psi_n(1/4)G_1(\sigma(s),s)q(s)f\Big(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\Big)\Delta s \\
&\ge \psi_n(1/4)\int_\nu^\xi G_1(\sigma(s),s)q(s)f\Big(s, \int_\alpha^\nu y(\tau)\Delta\tau\Big)\Delta s \\
&\ge \psi_n(1/4)\int_\nu^\xi G_1(\sigma(s),s)q(s)\mu_1\Big(\int_\alpha^\nu y(\tau)\Delta\tau\Big)\Delta s \\
&\ge \psi_n(1/4)(\nu-\alpha)\gamma_n\overline{\rho_*}\mu_1 \int_\nu^\xi G_1(\sigma(s),s)q(s)\Delta s \\
&\ge \overline{\rho_*} = \|y\|,
\end{aligned}$$

and so

$$\|Ty\| \ge \|y\| \;\text{ for all }\; y \in P \cap \partial\Omega_1. \tag{3.19}$$

Since $\lim_{y\to\infty} \dfrac{f(t,y)}{y} = \infty$, there exists $\rho^* > \rho_1$ such that

$$f(t,y) \ge \eta y \;\text{ for }\; y \ge \rho^*, \tag{3.20}$$

where $\mu_2 \geq \mu$; here μ is given in the proof of Theorem 3.3. Choose

$$\overline{\rho^*} > \max\left\{\frac{\rho^*}{(\nu-\alpha)\gamma_n}, \rho_1\right\}$$

and set $\Omega_2 = \{y \in B : \|y\| < \overline{\rho^*}\}$. For any $y \in P \cap \partial\Omega_2$, we get

$$\int_\alpha^\nu y(\tau)\Delta\tau \geq \int_\alpha^\nu \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} y(t)\Delta\tau \geq (\nu-\alpha)\gamma_n\|y\| = (\nu-\alpha)\gamma_n\overline{\rho^*} > \rho^*. \quad (3.21)$$

Using Lemma 2.1, (C_2), (3.20) and (3.21), for $t_0 \in [\alpha, \beta_n]_{\mathbb{T}}$, we have

$$\begin{aligned} Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0,s)q(s)f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &\geq \int_\alpha^\xi \psi_n(1/4)G_1(\sigma(s),s)q(s)f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &\geq \psi_n(1/4)\int_\nu^\xi G_1(\sigma(s),s)q(s)f\left(s, \int_\alpha^\nu y(\tau)\Delta\tau\right)\Delta s \\ &\geq \psi_n(1/4)\int_\nu^\xi G_1(\sigma(s),s)q(s)\mu_1\left(\int_\alpha^\nu y(\tau)\Delta\tau\right)\Delta s \\ &\geq \psi_n(1/4)(\nu-\alpha)\gamma_n\overline{\rho^*}\mu_1\int_\nu^\xi G_1(\sigma(s),s)q(s)\Delta s \\ &\geq \overline{\rho^*} = \|y\|, \end{aligned}$$

which yields

$$\|Ty\| \geq \|y\| \text{ for all } y \in P \cap \partial\Omega_2. \quad (3.22)$$

Let $\Omega_3 = \{y \in B : \|y\| < \rho_1\}$. For $y \in P \cap \partial\Omega_3$ from (C_6), we obtain

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s)q(s)f\left(s, \int_a^{\sigma(s)} y(\tau)\Delta\tau\right)\Delta s \\ &\leq \int_a^{\sigma(b)} \theta_n G_1(\sigma(s),s)q(s)f\left(s, \int_a^{\sigma(b)} y(\tau)\Delta\tau\right)\Delta s \\ &\leq \theta_n\int_a^{\sigma(b)} G_1(\sigma(s),s)q(s)\Gamma\rho_1\Delta s \\ &\leq \theta_n\Lambda\Gamma\rho_1\int_a^{\sigma(b)} G_1(\sigma(s),s)\Delta s \\ &\leq \theta_n\Lambda s_0\Gamma\rho_1 = \rho_1 = \|y\|, \end{aligned}$$

which yields

$$\|Ty\| \leq \|y\| \text{ for all } y \in P \cap \partial\Omega_3. \quad (3.23)$$

Hence, since $\rho_* \leq \rho_1 < \rho^*$ and from (3.19), (3.22) and (3.23) it follows from Theorem 2.6 that T has a fixed point y_1 in $P \cap (\overline{\Omega_3} \backslash \Omega_1)$ and a fixed point y_2 in $P \cap (\overline{\Omega_2} \backslash \Omega_3)$. Note both are positive solutions of the LBVP (3.4) satisfying

$$0 < \|y_1\| \leq \rho_1 < \|y_2\|.$$

From (3.3), two positive solutions of (1.1) are x_1 and x_2 such that $x_1(t) = \displaystyle\int_a^t y_1(s)\Delta s$ and $x_2(t) = \displaystyle\int_a^t y_2(s)\Delta s$. $\square$

Theorem 3.6. *Let* (C_1) *and* (C_2) *hold. Also assume*

(C_7) $\displaystyle\lim_{y\to 0^+} \frac{f(t,y)}{y} = 0,\ \lim_{y\to\infty} \frac{f(t,y)}{y} = 0,$ *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$.

(C_8) *There exists constant* ρ_2 *such that* $f(t,y) \geq \Theta\rho_2$, $y \in [(\nu-\alpha)\gamma_n\rho_2, (\nu-\alpha)\rho_2]$, *where* $\Theta = \left(\psi_n(1/4)\displaystyle\int_\nu^\xi G_1(\sigma(s),s)q(s)\Delta s\right)^{-1}$.

Then the LBVP (3.4) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| \leq \rho_2 < \|y_2\|.$$

and the CLBVP (1.1) *at least two positive solutions* x_1 *and* x_2 *with* $x_1(t) = \displaystyle\int_a^t y_1(s)\Delta s$ *and* $x_2(t) = \displaystyle\int_a^t y_2(s)\Delta s$.

Example 3.7. Let $\mathbb{T} = \mathbb{Z}$. We consider the following complementary Lidstone boundary value problem on $\mathbb{T}$:

$$\begin{aligned} &x^{\Delta^{(5)}}(t) + f(t, x^\sigma(t)) = 0,\ \ t \in [0,7]_{\mathbb{T}}, \\ &x(0) = 0,\ \ x^\Delta(0) = x^\Delta(\sigma^4(7)) = 0,\ \ x^{\Delta^{(3)}}(0) = x^{\Delta^{(3)}}(\sigma^2(7)) = 0. \end{aligned} \tag{3.24}$$

Note (3.24) is a particular case of (1.1) with $q(t) = 1$ and $a = 0, b = 7, n = 2$. Since $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t+1$, $\sigma^j(t) = t+j$ and $x^\Delta(t) = \Delta x(t)$, $x^{\Delta^{(j)}}(t) = \Delta^j x(t)$. We notice that our complementary Lidstone boundary value problem is the following difference complementary Lidstone boundary value problem:

$$\begin{aligned} &\Delta^5 x(t) + f(t, x(t+1)) = 0,\ \ t = 0,1,\dots,7 \\ &x(0) = 0,\ \ \Delta x(0) = \Delta x(11) = 0,\ \ \Delta^3 x(0) = \Delta^3 x(9) = 0. \end{aligned}$$

The Green's function $G_2^1(t,s)$ is

$$G_2^1(t,s) = \int_0^{\sigma^3(7)} G_2(t,r)G_1^1(r,s)\Delta r = \sum_{r=0}^{9} G_2(t,r)G_1^1(r,s),$$

where

$$\begin{aligned} G_2(t,s) &= \frac{-1}{\sigma^4(7)}\begin{cases} t(\sigma^4(7)-\sigma(s)),\ t\le s,\\ \sigma(s)(\sigma^4(7)-t),\ \sigma(s)\le t\end{cases}\\ &= \frac{-1}{11}\begin{cases} t(10-s), & t\le s,\\ (s+1)(11-t), & s+1\le t\end{cases}\end{aligned}$$

and

$$\begin{aligned} G_1^1(t,s)=G_1(t,s) &= \frac{-1}{\sigma^2(7)}\begin{cases} t(\sigma^2(7)-\sigma(s)), & t\le s,\\ \sigma(s)(\sigma^2(7)-t), & \sigma(s)\le t\end{cases}\\ &= \frac{-1}{9}\begin{cases} t(8-s), & t\le s,\\ (s+1)(9-t), & s+1\le t.\end{cases}\end{aligned}$$

In Lemma 2.1, we find $\theta_2 = \dfrac{s_2}{\sigma^4(7)\sigma^2(7)} = \dfrac{s_2}{11.9}$, where

$$\begin{aligned} s_2 &= \frac{1}{6}\Bigg\{(\sigma^4(7))^3+\sum_{t\in A_2}\mu(t)^2[3(\sigma^4(7))-2(t+2\sigma(t))]\\ &\quad -\sum_{t\in B_2}\mu(t)^2[2(t+2\sigma(t))-3(\sigma^4(7))]\Bigg\}\\ &= \frac{1}{6}\Bigg\{11^3+\sum_{t=0}^{4}29-6t-\sum_{t=5}^{10}6t-29\Bigg\}=220,\end{aligned}$$

with $A_2=\{0,1,2,3,4\}$ and $B_2=\{5,\dots,10\}$. In Lemma 2.3, we note $\alpha=1$, $\beta_2=10$, $\xi=8$, $\nu=4$ and we have

$$S_2 = \frac{1}{6}\Bigg\{1269+\sum_{t=1}^{4}29-6t-\sum_{t=5}^{9}6t-29\Bigg\}=210$$

and

$$\psi_2(\delta)=\delta^2\frac{S_2}{11\times 9},\gamma_2=\frac{210}{16\times 220}.$$

For condition (C_2), we note

$$0<\int_\nu^\xi G_1(\sigma(s),s)q(s)\Delta s=\int_4^8\sigma(s)(\sigma^2(7)-\sigma(s))\Delta s=\sum_{s=4}^{8}(s+1)(8-s)=60.$$

(i) Consider the complementary Lidstone dynamic equation (3.24) with the function $f(t,x)=x^2(t+x)$. It is easy to see that f satisfies condition (C_2). Since

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\lim_{y\to 0^+}\frac{y^2(t+y)}{y}=0,$$

$$\lim_{y\to\infty}\frac{f(t,y)}{y}=\lim_{y\to\infty}\frac{y^2(t+y)}{y}=\infty,$$

for $t \in [0,8]_{\mathbb{T}}$, condition (C_3) is fulfilled. Therefore, according to Theorem 3.3 the complementary Lidstone BVP (3.24) has at least one positive solution.

(ii) Consider the complementary Lidstone dynamic equation (3.24) with the function $f(t,x)=\sqrt{x}+t^2$. It is easy to see that f satisfies condition (C_2). Also we obtain

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\lim_{y\to 0^+}\frac{\sqrt{y}+t^2}{y}=\infty,$$

$$\lim_{y\to\infty}\frac{f(t,y)}{y}=\lim_{y\to\infty}\frac{\sqrt{y}+t^2}{y}=0,$$

for $t \in [0,8]_{\mathbb{T}}$, so condition (C_4) is fulfilled. From Theorem 3.4, the complementary Lidstone BVP (3.24) has at least one positive solution.

(iii) Consider the complementary Lidstone dynamic equation (3.24) with the function

$$f(t,x)=\begin{cases}\dfrac{5x^3}{32\times 1052(1+x)}, & x\geq 4;\\ \dfrac{\sqrt{x}}{1052}, & 0\leq x<4.\end{cases}$$

The function f is continuous on $[0,8]_{\mathbb{T}}\times\mathbb{R}$ and nondecreasing in the second argument with $f(t,x)\geq 0$ for $(t,x)\in[0,8]_{\mathbb{T}}\times K$. Hence condition ($C_2$) is fulfilled. Also we have

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\lim_{y\to 0^+}\frac{\sqrt{y}}{1052y}=\infty,$$

$$\lim_{y\to\infty}\frac{f(t,y)}{y}=\lim_{y\to\infty}\frac{5y^3}{32\times 1052(1+y)y}=\infty,$$

for $t\in[0,8]_{\mathbb{T}}$. Thus (C_5) is satisfied. Furthermore, we note $\Lambda=1$ and

$$\begin{aligned} s_0 &= \frac{1}{6}\Big\{(\sigma^2(7))^3+\sum_{t\in A_0}\mu(t)^2[3(\sigma^2(7))-2(t+2\sigma(t))] \\ &\quad -\sum_{t\in B_0}\mu(t)^2[2(t+2\sigma(t))-3(\sigma^1(7))]\Big\} \\ &= \frac{1}{6}\Big\{9^3+\sum_{t=0}^{3}23-6t-\sum_{t=4}^{9}6t-23\Big\}=120,\end{aligned}$$

and $\Gamma=\dfrac{1}{\theta_2\Lambda s_0}=\dfrac{99}{210\times 120}$. If we choose $\rho_1=\dfrac{1}{2}$, and noting f is nondecreasing, then we have

$$f(t,y)=\frac{\sqrt{y}}{1052}\leq 1,\ \text{for } 0\leq y\leq 4,\ t\in[0,8]_{\mathbb{T}}$$

so condition (C_6) is satisfied. Thus all the conditions of Theorem 3.5 are satisfied so the CLBVP (3.24) has at least two positive solutions.

(iv) Consider the complementary Lidstone dynamic equation (3.24) with the function

$$f(t,x) = \begin{cases} \sqrt{x-1}+7, & x \geq 1; \\ \dfrac{14x^2}{1+x}, & 0 \leq x < 1. \end{cases}$$

The function f is continuous on $[0,8]_{\mathbb{T}} \times \mathbb{R}$ and nondecreasing in the second argument with $f(t,x) \geq 0$ for $(t,x) \in [0,8]_{\mathbb{T}} \times K$. Hence condition ($C_2$) is fulfilled. Also we have

$$\lim_{y\to 0^+} \frac{f(t,y)}{y} = \lim_{y\to 0^+} \frac{14y^2}{(1+y)y} = 0,$$

$$\lim_{y\to\infty} \frac{f(t,y)}{y} = \lim_{y\to\infty} \frac{\sqrt{y-1}+7}{y} = 0,$$

for $t \in [0,8]_{\mathbb{T}}$. Thus (C_7) is satisfied. Now if we calculate the number Θ in Theorem 3.6, then we obtain $\Theta = \dfrac{1}{\psi_2(1/4)60} = \dfrac{1}{350}$. If we choose $\rho_2 = \dfrac{1}{3}$, and noting f is nondecreasing, then we have

$$f(t,y) = \frac{14y^2}{1+y} \geq \frac{11}{350} \times \frac{1}{3}, \text{ for } \frac{21}{16\times 22} \leq y \leq 1, \ t \in [0,8]_{\mathbb{T}}$$

so condition (C_8) is satisfied. Thus all the conditions of Theorem 3.6 are satisfied so the CLBVP has at least two positive solutions.

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