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Existence, Uniqueness, and Convergence of the Optimal Derivative for a Class of Nonlinear Functions

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Abstract

In this paper, we study the so-called optimal derivative for a specific class of nonlinear functions. Results on the existence, uniqueness, and convergence of the approximation are presented.

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1 Introduction

The study of differential equations is a mathematical field that has historically been the subject of much research, however, continues to remain relevant, by the fact that it is of particular interest in such disciplines as engineering, physical sciences and more

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recently biology and electronics, in which many models lead to equations of the same type. Most of these equations are generally nonlinear in nature. The term "nonlinear" gathers extremely diverse systems with little in common in their behavior. It follows that there is not, so far, a theory of nonlinear equations. A large class of these nonlinear problems is modelled by nonlinear ordinary differential equations.

Consider the nonregular case. Imagine the case when the only equilibrium point is nonregular. In this case, we cannot derive the nonlinear function and consequently we cannot study the linearized equation. A natural question arises then: Is it possible to associate another linear equation to the nonlinear equation which has the same asymptotic behavior?

The idea proposed by Benouaz and Arino is based on the method of approximation. In [3, 4, 7–9], the authors introduced the optimal derivative, which is in fact a global approximation as opposed to the nonlinear perturbation of a linear equation, having a distinguished behavior with respect to the classical linear approximation in the neighborhood of the stationary point. The approach used is the least square approximation. Benouaz and Bohner have developed the optimal derivative, in particular, the relationship between the optimal derivative and the classical linearization [10] and the relationship between the optimal derivative and asymptotic stability [11] (for applications see also [1, 12]).

The aim of this paper is to make some theoretical progress and investigate the validity of method. In this paper, we study a specific class of nonlinear functions. We present results on the existence, uniqueness, and convergence of the approximation. Related versions of these results are contained in the PhD thesis of the first author, see [2, 5]. At first, we present a brief overview of the optimal derivative.

2 Optimal Derivative Review

2.1 The Procedure

Consider a nonlinear ordinary differential problem of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x), \quad x(0) = x_0, \tag{2.1}$$

where

- $x = (x_1, \dots, x_n)$ is the unknown function,
- $F = (f_1, \dots, f_n)$ is a given function on an open subset $\Omega \subset \mathbb{R}^n$,

with the assumptions

(H$_1$) $F(0) = 0$,

(H_2) the spectrum $\sigma(DF(x))$ is contained in the set $\{z : \mathrm{Re} z < 0\}$ for every $x \neq 0$, in a neighborhood of 0 for which $DF(x)$ exists,

(H_3) F is γ-Lipschitz continuous.

Consider $x_0 \in \mathbb{R}^n$ and the solution x of the nonlinear equation starting at x_0. With all linear $A \in \mathcal{L}(\mathbb{R}^n)$, we associate the solution y of the problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay(t), \quad y(0) = y_0,$$

and we try to minimize the functional

$$G(A) = \int_0^\infty \|F(y(t)) - Ay(t)\|^2 \,\mathrm{d}t \tag{2.2}$$

along a solution y. We obtain

$$\widetilde{A} = \left(\int_0^\infty F(x(t))[x(t)]^T \mathrm{d}t\right)\left(\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right)^{-1}. \tag{2.3}$$

Precisely, the procedure is defined by the following scheme: Given x_0, we choose a first linear map. For example, if F is differentiable in x_0, then we can take $A_0 = DF(x_0)$ or the derivative value in a point in the vicinity of x_0. This is always possible if F is locally Lipschitz. If A_0 is an asymptotically stable map, then the solution starting from x_0 of the problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A_0 y(t), \quad y(0) = y_0$$

tends to 0 exponentially. We can evaluate $G(A)$ using (2.2) and we minimize G for all matrices A. If F is linear, then the minimum is reached for the value $A = F$ (and we have $A_0 = F$). Generally, we can always minimize G, and the matrix which gives the minimum is unique. We call this matrix A_1 and replace A_0 by A_1, we replace y by the solution of the linearized equation associated to A_1, and we continue. The optimal derivative $\widetilde{A}$ is given by (2.3) and is the limit of the sequence build as such (for details see [3, 6–8]).

2.2 Properties of the Procedure

We now consider situations where the procedure converges.

Influence of the choice of the initial condition

Note that if we change $x(t)$ to z, then the relation (2.3) can be written as

$$\widetilde{A} \oint_0^{x_0} z \mathrm{d}z^T = \oint_0^{x_0} F(z) \mathrm{d}z^T,$$

where $\oint_0^{x_0}$ is the curvilinear integral along the orbit $\gamma(x_0) = \{e^{tB} : t \geq 0\}$ of x_0. We obtain

$$\widetilde{A} = \left(\oint_0^{x_0} F(z) \mathrm{d} z^T \right) \left(\oint_0^{x_0} z \mathrm{d} z^T \right)^{-1}.$$

It is clear that the optimal derivative depends on the initial condition x_0.

Case when F is linear

If F is linear with $\sigma(F)$ inside the left-hand side of the complex plane, then the procedure gives F at the first iteration. Indeed, in this case, (2.3) reads

$$A\Gamma(x) = F\Gamma(x)$$

and it is clear that $A = F$ is a solution. It is unique if $\Gamma(x)$ is invertible. Therefore, the optimal approximation of a linear system is the system itself.

Case when F is the sum of a linear and nonlinear term

Consider the more general system of nonlinear equations with a nonlinearity of the form

$$F(x) = Mx + \widetilde{F}(x), \quad x(0) = x_0,$$

where M is linear. The computation of the matrix A_1 gives

$$\begin{aligned} A_1 &= \left[\int_0^\infty F(x(t))[x(t)]^T \mathrm{d}t \right] [\Gamma(x)]^{-1} \\ &= \left(M\Gamma(x) + \int_0^\infty \widetilde{F}(x(t))[x(t)]^T \mathrm{d}t \right) [\Gamma(x)]^{-1} \\ &= M + \left(\int_0^\infty \widetilde{F}(x(t))[x(t)]^T \mathrm{d}t \right) [\Gamma(x)]^{-1}. \end{aligned}$$

Hence, $A_1 = M + \widetilde{A}_1$ with

$$\widetilde{A}_1 = \left(\int_0^\infty \widetilde{F}(x(t))[x(t)]^T \mathrm{d}t \right) [\Gamma(x)]^{-1}.$$

Then, for all j we have $A_j = M + \widetilde{A}_j$ with

$$\widetilde{A}_j = \left(\int_0^\infty \widetilde{F}(x_j(t))[x_j(t)]^T \mathrm{d}t \right) [\Gamma(x_j)]^{-1}.$$

If, in particular, some components of F are linear, then the corresponding components of $\widetilde{F}$ are zero, and the corresponding components of A_j are those of F. If f_k is linear, then the kth row of the matrix A_j is equal to f_k.

2.3 Calculation of $\widetilde{A}$

Let us suppose that the sequence A_j given by

$$A_j = \left(\int_0^\infty F\left(e^{tA_{j-1}}x_0\right)\left[e^{tA_{j-1}}x_0\right]^T \mathrm{d}t\right)\left(\int_0^\infty e^{tA_{j-1}}x_0\left[e^{tA_{j-1}}x_0\right]^T \mathrm{d}t\right)^{-1}$$

converges to the optimal matrix and that the derivative $DF(0)$ of F in 0 exists. In this case, we can write

$$F(x) = DF(0)x + \mathrm{o}(|x|). \tag{2.4}$$

Replacing the relation (2.4) and using the properties of the optimal derivative from [7,8], we find

$$\begin{aligned}
\widetilde{A} &= \left[\int_0^\infty [DF(0)x(t) + \mathrm{o}(|x(t)|)]\,[x(t)]^T \mathrm{d}t\right]\left[\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right]^{-1} \\
&= DF(0)\left[\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right]\left[\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right]^{-1} \\
&\quad + \left[\int_0^\infty \mathrm{o}(|x(t)|)[x(t)]^T \mathrm{d}t\right]\left[\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right]^{-1} \\
&= DF(0) + \left[\int_0^\infty \mathrm{o}(|x(t)|)[x(t)]^T \mathrm{d}t\right]\left[\int_0^\infty x(t)[x(t)]^T \mathrm{d}t\right]^{-1},
\end{aligned}$$

where

$$\left[\int_0^\infty [\mathrm{o}(|x(t)|)][x(t)]^T \mathrm{d}t\right]\left[\int_0^\infty [x(t)][x(t)]^T \mathrm{d}t\right]^{-1} = \mathrm{o}(1),$$

i.e., a quantity which tends to 0 when $x_0 \to 0$, by supposing that $|x(t)|$ remains of the order of x_0.

Remark 2.1. It should be noted that the nonlinear system is written after application of the optimal linearization as

$$\widetilde{A} = M + r(x_0), \quad x_0 = x(0),$$

where $M = DF(0)$ and

$$r(x_0) = \left[\int_0^\infty G\left(e^{tA}x_0\right)\left[e^{tA}x_0\right]^T \mathrm{d}t\right]\left[\int_0^\infty e^{tA}x_0\left[e^{tA}x_0\right]^T \mathrm{d}t\right]^{-1}.$$

The first term is the linearization. The second term, which is actually the optimal linearization of the nonlinear function G, turns out to be dependent on the initial value x_0. It is as if we had perturbed $DF(0)$, writing the optimal matrix in the form

$$\widetilde{A} = DF(0) + \mathrm{o}\left(\|x_0\|\right).$$

3 A Special Class of Nonlinear Functions

3.1 Theoretical Framework

We fix $x_0 \neq 0$ and define in $M_n(\mathbb{R})$ the map $A \to \Phi(A)$ such that

$$\Phi(A) = \left[\int_0^\infty F\left(e^{tA}x_0\right)\left[e^{tA}x_0\right]^T \mathrm{d}t\right][\Gamma(A)]^{-1} \tag{3.1}$$

with

$$\Gamma(A) = \int_0^\infty \left(e^{tA}x_0\right)\left[e^{tA}x_0\right]^T \mathrm{d}t.$$

Now we build the sequence of approximations

$$\begin{cases} A_j = \Phi\left(A_{j-1}\right) \\ A_0 = DF(x_0), \end{cases} \tag{3.2}$$

where x_0 is an arbitrary point taken in a neighborhood of 0 and such that F is differentiable at this point. The limit of the sequence A_j, if it exists, is called the fixed point optimal derivative. From this, we can obtain a linear optimal equation. In fact, it is a fixed-point problem that we are going to solve. Indeed, we have in case of convergence of the sequence A_j the existence and uniqueness if we can show that Φ is Lipschitz continuous with Lipschitz constant less than 1.

3.2 Problem Setting

Define

$$\triangle = \begin{bmatrix} \delta_1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & \delta_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & 0 & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & \delta_{n-1} & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & \delta_n \end{bmatrix}, \quad 0 < \delta_1 < \delta_2 < \ldots < \delta_n. \tag{3.3}$$

We consider a nonlinear ordinary differential problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = F(x) \\ x(0) = x_0, \end{cases} \tag{3.4}$$

where

$$F(x) = -\triangle x + G(x), \quad x \in \mathbb{R}^n.$$

F is defined in an open set Ω with values in $\mathbb{R}^n$. We assume that $F(0) = 0$, F is continuous, and F is Lipschitz. x_0 is chosen with nonzero components. We are interested in the existence, the uniqueness, and the convergence of the optimal derivative obtained by minimization of the "lesser square", associated to the equation (3.4) of the form

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \widetilde{A}x \\ x(0) = x_0. \end{cases} \tag{3.5}$$

$\widetilde{A}$ is the limit, if it exists, of the sequence defined by (3.2).

3.3 Recursive Relation

By using the expression of F, the recursive relation between A_{j-1} and A_j is

$$\begin{aligned} A_j &= -\triangle \left(\int_0^\infty e^{tA_{j-1}} x_0 \left[e^{tA_{j-1}} x_0 \right]^T \mathrm{d}t \right) \left[\Gamma\left(A_{j-1}\right) \right]^{-1} \\ &\quad + \left(\int_0^\infty G\left(e^{tA_{j-1}} x_0\right) \left[e^{tA_{j-1}} x_0 \right]^T \mathrm{d}t \right) \left[\Gamma\left(A_{j-1}\right) \right]^{-1} \\ &= -\triangle + \left(\int_0^\infty G\left(e^{tA_{j-1}} x_0\right) \left[e^{tA_{j-1}} x_0 \right]^T \mathrm{d}t \right) \left[\Gamma\left(A_{j-1}\right) \right]^{-1}, \end{aligned}$$

where Γ is invertible. If we set

$$\varphi(A) = \left(\int_0^\infty \left[G\left(e^{tA} x_0\right) \right] \left[e^{tA} x_0 \right]^T \mathrm{d}t \right) [\Gamma(A)]^{-1} \tag{3.6}$$

and

$$\Phi(A) = -\triangle + \varphi(A), \tag{3.7}$$

then we obtain the problem

$$\begin{cases} A_j = \Phi\left(A_{j-1}\right) \\ A_0 \in B(-\triangle, \rho), \end{cases} \tag{3.8}$$

where $B(-\triangle, \rho)$ is the ball with center $-\triangle$ and radius $\rho > 0$ in the space of matrices $M_n(\mathbb{R})$. The initial matrix is

$$A_0 = -\triangle + DG(x_0). \tag{3.9}$$

4 Existence of the Optimal Derivative

Lemma 4.1. *Assume* (3.3). *Let* $\gamma > 0$ *and suppose* G *is continuous and satisfies*

$$\|G(x)\| \leq \gamma \|x\| \quad \textit{for all} \quad x \in \mathbb{R}^n. \tag{4.1}$$

If $B \in B(-\triangle, \rho)$ for some $\rho \in (0, \delta_1/2)$, then there exists a constant $m = m(x_0, \triangle)$ such that φ defined by (3.6) *satisfies*

$$\|\varphi(B)\| \le K, \quad \textit{where} \quad K = \frac{\gamma\left(\frac{\delta_n-\delta_1}{2}+2\rho\right)^2\|x_0\|^2}{2m(\delta_1-2\rho)\rho^2}. \tag{4.2}$$

Proof. From (4.1), we have

$$\|\varphi(B)\| \le \gamma\left(\int_0^\infty \left\|e^{tB}x_0\right\|^2 \mathrm{d}t\right)\left\|[\Gamma(B)]^{-1}\right\|. \tag{4.3}$$

We first estimate $\left\|[\Gamma(B)]^{-1}\right\|$. To do this, we compute

$$v^T\Gamma(B)v = \int_0^\infty \left(v^T e^{tB}x_0\right)^2 \mathrm{d}t, \tag{4.4}$$

and so $v^T\Gamma(B)v$ is the integral of a square function. So, $v^T\Gamma(B)v$ is nonzero if the function is nonzero. We take $B = -\triangle$, and the square of the function which we compute is

$$\psi(v(t)) = \sum_{j=1}^n v_j x_{0,j}\, e^{-\delta_j t},$$

where $0 < \delta_1 < \delta_2 < \ldots < \delta_n$ and $\psi(v(t))$ is equivalent, when $t \to \infty$, to a term which corresponds to the smallest integer j such that

$$v_j x_{0,j} \neq 0, \quad \text{which means} \quad v_j \neq 0,$$

so we have $v^T\Gamma(B)v > 0$ for all $v \neq 0$. Hence the quadratic form associated with $\Gamma(B)$ is positive definite, and so for all $(x_0, \triangle)$, there exists a constant $m = m(x_0, \triangle) > 0$ such that

$$v^T\Gamma(B)v \ge m\|v\|^2. \tag{4.5}$$

Thus

$$\|\Gamma(B)\| \ge m. \tag{4.6}$$

With $\lambda_{\min} = \min\{\lambda : \lambda \in \sigma(\Gamma(B))\}$, (4.6) yields

$$\lambda_{\min} \ge m$$

which implies for all $\lambda \in \sigma\left([\Gamma(B)]^{-1}\right)$, $\lambda \le \lambda_{\min}^{-1}$ and

$$\lambda \le \frac{1}{m}. \tag{4.7}$$

Thus

$$v^T[\Gamma(B)]^{-1}v \le \frac{1}{m}\|v\|^2. \tag{4.8}$$

We deduce that the positive definite symmetric matrix $[\Gamma(B)]^{-1}$ satisfies

$$\left\|[\Gamma(B)]^{-1}\right\| \leq \frac{1}{m}. \tag{4.9}$$

Now we estimate the term $\left\|e^{tB}\right\|$. We define C to be the circle in the left-hand side of the complex plane with

$$\text{center at} \quad \left(-\frac{\delta_1+\delta_n}{2}, 0\right) \quad \text{and radius} \quad \frac{\delta_n-\delta_1}{2}+2\rho. \tag{4.10}$$

Then C encloses the numbers $-\delta_i$, $1 \leq i \leq n$, and also the eigenvalues of B for

$$\|B+\triangle\| < \rho < \frac{\delta_1}{2}.$$

Moreover,

$$\delta_C := \operatorname{dist}\left(C, \{-\delta_i\}_{1\leq i\leq n}\right) = 2\rho.$$

Then by Cauchy's integral formula,

$$e^{tB} = \frac{1}{2\pi i}\int_C e^{tz}(zI-B)^{-1}\mathrm{d}z, \tag{4.11}$$

and if we put $\eta = \delta_1 - 2\rho$, then

$$\left\|e^{tB}\right\| \leq \frac{1}{2\pi}\int_C \left|e^{tz}\right| \left\|(zI-B)^{-1}\right\| |\mathrm{d}z| \leq \frac{1}{2\pi}\int_C e^{-t\eta}\left\|(zI-B)^{-1}\right\| |\mathrm{d}z|.$$

Let $z \in C$. Note that

$$(zI-B)^{-1} = \left(I-(zI+\triangle)^{-1}(B+\triangle)\right)^{-1}(zI+\triangle)^{-1} \tag{4.12}$$

holds. We calculate

$$\begin{aligned}\left\|(zI+\triangle)^{-1}\right\| &= \max_{1\leq i\leq n}\frac{1}{|z+\delta_i|} = \frac{1}{\min_{1\leq i\leq n}|z+\delta_i|}\\ &= \frac{1}{\operatorname{dist}\left(z,\{-\delta_i\}_{1\leq i\leq n}\right)} \leq \frac{1}{\delta_C} = \frac{1}{2\rho}.\end{aligned}$$

Since

$$\left\|(zI+\triangle)^{-1}(B+\triangle)\right\| \leq \left\|(zI+\triangle)^{-1}\right\|\cdot\|B+\triangle\| \leq \frac{\|B+\triangle\|}{2\rho} < \frac{1}{2} < 1,$$

we have (see, e.g., [14, exercise following Corollary 5.6.16])

$$\begin{aligned}\left\|\left(I-(zI+\triangle)^{-1}(B+\triangle)\right)^{-1}\right\| &\leq \frac{1}{1-\|(zI+\triangle)^{-1}(B+\triangle)\|}\\ &\leq \frac{1}{1-\|(zI+\triangle)^{-1}\|\cdot\|B+\triangle\|}.\end{aligned}$$

Using this in (4.12), we find

$$\left\|(zI-B)^{-1}\right\| \leq \left\|\left(I-(zI+\triangle)^{-1}(B+\triangle)\right)^{-1}\right\|\cdot\left\|(zI+\triangle)^{-1}\right\|$$

$$
\begin{aligned}
&\le \frac{\|(zI+\triangle)^{-1}\|}{1-\|(zI+\triangle)^{-1}\|\cdot\|B+\triangle\|}\\
&= \frac{1}{\frac{1}{\|(zI+\triangle)^{-1}\|}-\|B+\triangle\|}\\
&\le \frac{1}{2\rho-\|B+\triangle\|} < \frac{1}{2\rho-\rho} = \frac{1}{\rho}.
\end{aligned}
\tag{4.13}
$$

Thus

$$
\left\|e^{tB}\right\| \le \frac{ke^{-t\eta}}{\rho}, \quad \text{where} \quad k = \frac{1}{2\pi}\int_C |\mathrm{d}z| = \frac{\delta_n-\delta_1}{2}+2\rho. \tag{4.14}
$$

Putting (4.9) and (4.14) in (4.3), we obtain

$$
\|\varphi(B)\| \le \frac{\gamma k^2\|x_0\|^2}{m\rho^2}\int_0^\infty e^{-2t\eta}\mathrm{d}t = \frac{\gamma k^2\|x_0\|^2}{2m\eta\rho^2}
$$

which shows (4.2) and completes the proof. □

Proposition 4.2. *Under the assumptions of Lemma 4.1, there exists $\rho > 0$ such that the equation*

$$
B = -\triangle + \varphi(B) \tag{4.15}
$$

has at least one solution in $B(-\triangle, \rho)$.

Proof. If we put $K < \rho$, where K is defined in (4.2), then we have $\|\varphi(B)\| < \rho$. Then the map Φ defined by (3.7) satisfies

$$
\Phi : B(-\triangle,\rho) \to B(-\triangle,\rho).
$$

This map is continuous. The continuity of Φ is a consequence of the continuity of each of maps

$$
B \to [\Gamma(B)]^{-1} \quad \text{and} \quad B \to \int_0^\infty G(x(t))\left[e^{tB}x_0\right]^T\mathrm{d}t.
$$

Indeed, for the first map, it suffices to show the continuity of the map $B \to \Gamma(B)$, since we know that $\Gamma(B)$ is uniformly invertible in a neighborhood of $B = 0$. By noting that

$$
\Gamma(B) = \int_0^\infty x(t)[x(t)]^T\mathrm{d}t = \int_0^\tau x(t)[x(t)]^T\mathrm{d}t + \int_\tau^\infty x(t)[x(t)]^T\mathrm{d}t,
$$

we see that the integral from 0 to τ depends continuously on B, and the integral from τ to ∞ is uniformly small when τ is big enough. In the same manner, for the second map, by noting that

$$
\psi(B) = \int_0^\infty G(x(t))[x(t)]^T\mathrm{d}t = \int_0^\tau G(x(t))[x(t)]^T\mathrm{d}t + \int_\tau^\infty G(x(t))[x(t)]^T\mathrm{d}t,
$$

we see that the integral from 0 to τ depends continuously on B, and the integral from τ to ∞ is uniformly small when τ is big enough. So we may write

$$\Gamma(B) = \lim_{\tau\to\infty} \int_0^\tau e^{tB}x_0 \left[e^{tB}x_0\right]^T \mathrm{d}t,$$
$$\psi(B) = \lim_{\tau\to\infty} \int_0^\tau G(e^{tB}x_0) \left[e^{tB}x_0\right]^T \mathrm{d}t$$

uniformly, i.e., $\Gamma(B)$ and $\varphi(B)$ are uniform limits of continuous functions and are so continuous. Thus Φ applies the closed ball to itself and is continuous. From Brouwer's fixed point theorem [13], we conclude the existence of a fixed point of Φ. □

5 Uniqueness of the Optimal Derivative

Lemma 5.1. *Assume* (3.3)*. Let $\gamma > 0$ and suppose G satisfies $G(0) = 0$ and*

$$\|G(x_1) - G(x_2)\| \le \gamma \|x_1 - x_2\| \quad \textit{for all} \quad x_1, x_2 \in \mathbb{R}^n. \tag{5.1}$$

If $B_1, B_2 \in B(-\triangle, \rho)$ for some $\rho \in (0, \delta_1/2)$, then there exists $m = m(x_0, \triangle)$ such that φ defined by (3.6) *satisfies*

$$\|\varphi(B_2) - \varphi(B_1)\| \le \overline{K} \|B_2 - B_1\|,$$
$$\textit{where} \quad \overline{K} = \frac{\gamma\left(\frac{\delta_n-\delta_1}{2} + 2\rho\right)^2 \|x_0\|^2}{m(\delta_1 - 2\rho)\rho^3} \left(1 + \frac{\left(\frac{\delta_n-\delta_1}{2} + 2\rho\right)^2 \|x_0\|^2}{2m(\delta_1 - 2\rho)\rho^2}\right). \tag{5.2}$$

Proof. We first estimate the term $\left\|e^{tB_2} - e^{tB_1}\right\|$. We choose the circle C in the complex plane defined by (4.10). Then C encloses the numbers $-\delta_i$, $1 \le i \le n$, and also the eigenvalues of B_1 and B_2 for $\|B_1 + \triangle\| < \rho < \delta_1/2$ and $\|B_2 + \triangle\| < \rho$. Moreover, the distance from C to $\{-\delta_i\}_{1\le i\le n}$ is equal to 2ρ. As in the proof of Lemma 4.1, we put $\eta = \delta_1 - 2\rho$. As in (4.11), we obtain

$$\begin{aligned} e^{tB_2} - e^{tB_1} &= \frac{1}{2\pi i}\int_C e^{tz}(zI - B_2)^{-1}\mathrm{d}z - \frac{1}{2\pi i}\int_C e^{tz}(zI - B_1)^{-1}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_C e^{tz}\left[(zI - B_2)^{-1} - (zI - B_1)^{-1}\right]\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_C e^{tz}(zI - B_2)^{-1}\left[(zI - B_1) - (zI - B_2)\right](zI - B_1)^{-1}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_C e^{tz}(zI - B_2)^{-1}(B_2 - B_1)(zI - B_1)^{-1}\mathrm{d}z \end{aligned}$$

and therefore

$$\left\|e^{tB_2} - e^{tB_1}\right\| \le \frac{\|B_2 - B_1\|}{2\pi}\int_C e^{-t\eta}\left\|(zI - B_2)^{-1}\right\| \cdot \left\|(zI - B_1)^{-1}\right\| |\mathrm{d}z|$$

$$\leq \frac{\|B_2 - B_1\|}{2\pi}\frac{e^{-t\eta}}{\rho^2}\int_C |\mathrm{d}z| = \frac{ke^{-t\eta}\|B_2 - B_1\|}{\rho^2}, \tag{5.3}$$

where k is as in (4.14) and where we used the estimates

$$\left\|(zI - B_1)^{-1}\right\| \leq \frac{1}{\rho} \quad \text{and} \quad \left\|(zI - B_2)^{-1}\right\| \leq \frac{1}{\rho}$$

derived as in (4.13). Next, we calculate

$$\begin{aligned}
\varphi(B_2) - \varphi(B_1) &= \left[\int_0^\infty G\left(e^{tB_2}x_0\right)\left[e^{tB_2}x_0\right]^T \mathrm{d}t\right][\Gamma(B_2)]^{-1} \\
&\quad - \left[\int_0^\infty G\left(e^{tB_1}x_0\right)\left[e^{tB_1}x_0\right]^T \mathrm{d}t\right][\Gamma(B_1)]^{-1} \\
&= \left\{\int_0^\infty \left[G\left(e^{tB_2}x_0\right)\left[e^{tB_2}x_0\right]^T - G\left(e^{tB_1}x_0\right)\left[e^{tB_1}x_0\right]^T\right]\mathrm{d}t\right\}[\Gamma(B_2)]^{-1} \\
&\quad + \left[\int_0^\infty G\left(e^{tB_1}x_0\right)\left[e^{tB_1}x_0\right]^T \mathrm{d}t\right]\left[[\Gamma(B_2)]^{-1} - [\Gamma(B_1)]^{-1}\right] \\
&= \left\{\int_0^\infty \left[G\left(e^{tB_2}x_0\right) - G\left(e^{tB_1}x_0\right)\right]\left[e^{tB_2}x_0\right]^T \mathrm{d}t\right. \\
&\quad \left. + \int_0^\infty G\left(e^{tB_1}x_0\right)\left[e^{tB_2}x_0 - e^{tB_1}x_0\right]^T \mathrm{d}t\right\}[\Gamma(B_2)]^{-1} \\
&\quad + \left[\int_0^\infty G\left(e^{tB_1}x_0\right)\left[e^{tB_1}x_0\right]^T \mathrm{d}t\right][\Gamma(B_2)]^{-1}\left[\Gamma(B_1) - \Gamma(B_2)\right][\Gamma(B_1)]^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(B_1) - \Gamma(B_2) &= \int_0^\infty e^{tB_1}x_0x_0^T\left[e^{tB_1}\right]^T \mathrm{d}t - \int_0^\infty e^{tB_2}x_0x_0^T\left[e^{tB_2}\right]^T \mathrm{d}t \\
&= \int_0^\infty \left(e^{tB_1} - e^{tB_2}\right)x_0x_0^T\left[e^{tB_1}\right]^T \mathrm{d}t + \int_0^\infty e^{tB_2}x_0x_0^T\left[e^{tB_1} - e^{tB_2}\right]^T \mathrm{d}t.
\end{aligned}$$

Using here (4.14), i.e.,

$$\left\|e^{tB_1}\right\| \leq \frac{ke^{-t\eta}}{\rho} \quad \text{and} \quad \left\|e^{tB_2}\right\| \leq \frac{ke^{-t\eta}}{\rho},$$

where k is as in (4.14), (5.3), and (4.9), i.e.,

$$\left\|[\Gamma(B_1)]^{-1}\right\| \leq \frac{1}{m} \quad \text{and} \quad \left\|[\Gamma(B_2)]^{-1}\right\| \leq \frac{1}{m},$$

we find the estimate

$$\|\varphi(B_2) - \varphi(B_1)\| \leq \left\{\int_0^\infty \gamma\left\|e^{tB_2} - e^{tB_1}\right\| \cdot \|x_0\|^2 \cdot \left\|e^{tB_2}\right\| \mathrm{d}t\right.$$

$$
\begin{aligned}
& + \int_0^\infty \gamma \left\| e^{tB_1} \right\| \cdot \|x_0\|^2 \cdot \left\| e^{tB_2} - e^{tB_1} \right\| \mathrm{d}t \Bigg\} \frac{1}{m} \\
& \int_0^\infty \gamma \left\| e^{tB_1} \right\|^2 \cdot \|x_0\|^2 \, \mathrm{d}t \frac{1}{m^2} \left\{ \int_0^\infty \left\| e^{tB_2} - e^{tB_1} \right\| \cdot \|x_0\|^2 \cdot \left\| e^{tB_1} \right\| \mathrm{d}t \right. \\
& \left. + \int_0^\infty \left\| e^{tB_2} \right\| \cdot \|x_0\|^2 \cdot \left\| e^{tB_2} - e^{tB_1} \right\| \mathrm{d}t \right\} \\
\leq \; & \frac{\gamma k^2 \|x_0\|^2 \cdot \|B_2 - B_1\|}{2\eta m \rho^3} + \frac{\gamma k^2 \|x_0\|^2 \cdot \|B_2 - B_1\|}{2\eta m \rho^3} \\
& + \frac{\gamma k^2 \|x_0\|^2}{2\eta m^2 \rho^2} \left\{ \frac{k^2 \|x_0\|^2 \cdot \|B_2 - B_1\|}{2\eta \rho^3} + \frac{k^2 \|x_0\|^2 \cdot \|B_2 - B_1\|}{2\eta \rho^3} \right\},
\end{aligned}
$$

which shows (5.2) and completes the proof. □

Proposition 5.2. *Under the assumptions of Lemma 5.1, there exists $\rho > 0$ such that the equation* (4.15) *has exactly one solution in $B(-\triangle, \rho)$.*

Proof. If we put $\overline{K} < 1$, where $\overline{K}$ is defined in (5.2), then the map Φ defined by (3.7) is Lipschitz with Lipschitz constant smaller than 1 and satisfies (by the proof of Proposition 4.2)

$$\Phi : B(-\triangle, \rho) \to B(-\triangle, \rho).$$

We conclude that Φ has a unique fixed point. □

6 Convergence of the Optimal Derivative

Theorem 6.1. *Assume* (3.3). *Let $\gamma > 0$ and suppose G satisfies $G(0) = 0$ and* (5.1). *Then there exists $\rho > 0$ such that the map Φ defined by* (3.7) *is a strict contraction and satisfies*

$$\Phi : B(-\triangle, \rho) \to B(-\triangle, \rho).$$

Then the sequence A_j converges to the unique fixed point of Φ provided

$$A_0 \in B(-\triangle, \rho).$$

Proof. We have

$$A_0 \in B(-\triangle, \rho)$$

if and only if

$$A_0 + \triangle = DG(x_0) \in B(0, \rho).$$

From the relation (4.1), we have

$$\|DG(x_0)\| \leq \gamma.$$

If we impose the condition

$$\gamma < \rho,$$

then we have

$$A_0 \in B(-\triangle, \rho).$$

In this case, the sequence

$$\begin{cases} A_j = -\triangle + \Phi(A_{j-1}) \\ A_0 \in B(-\triangle, \rho) \end{cases} \tag{6.1}$$

converges to $\widetilde{A}$. In consequence, for all $A_0 \in B(-\triangle, \rho)$, the sequence A_j is well defined with values in $B(-\triangle, \rho)$, and it converges geometrically to the unique fixed point in the ball. □

References

[1] Sidi Mohammed Amine Bekkouche, Tayeb Benouaz, and Martin Bohner. Detection of the existence of bifurcation surfaces using the optimal derivative. *Int. J. Math. Comput.*, 11(J11):50–60, 2011.

[2] Tayeb Benouaz. *Contribution à l'approximation et la synthèse de la stabilité d'une equation diffèrentielle non linéaire*. PhD thesis, Université de Pau (France) and Université Tlemcen (Algérie), 1996.

[3] Tayeb Benouaz. Optimal derivative of a nonlinear ordinary differential equation. In *Equadiff 99, international conference on differential equations*, volume 2, pages 1404–1407. World Scientific Publishing Co. Pte. Ltd., 2000.

[4] Tayeb Benouaz and Ovide Arino. Determination of the stability of a non-linear ordinary differential equation by least square approximation. Computational procedure. *Appl. Math. Comput. Sci.*, 5(1):33–48, 1995.

[5] Tayeb Benouaz and Ovide Arino. Existence, unicité et convergence de l'approximation au sens des moindres — Carrés d'une equation différentielle ordinaire nonlinéaire. Number 94/19, Université de Pau, 1995. Publications de l'U.R.A. 1204 – CNRS.

[6] Tayeb Benouaz and Ovide Arino. Relation entre l'approximation optimale et la stabilité asymptotique. Number 95/10, Université de Pau, 1995. Publications de l'U.R.A. 1204 – CNRS.

[7] Tayeb Benouaz and Ovide Arino. Least square approximation of a nonlinear ordinary differential equation. *Comput. Math. Appl.*, 31(8):69–84, 1996.

[8] Tayeb Benouaz and Ovide Arino. Optimal approximation of the initial value problem. *Comput. Math. Appl.*, 36(1):21–32, 1998.

[9] Tayeb Benouaz and Fewzi Bendahmane. Least-square approximation of a nonlinear O.D.E. with excitation. *Comput. Math. Appl.*, 47(2-3):473–489, 2004.

[10] Tayeb Benouaz and Martin Bohner. On the relationship between the classical linearization and optimal derivative. *Adv. Dyn. Syst. Appl.*, 2(1):41–57, 2007.

[11] Tayeb Benouaz, Martin Bohner, and Abdelhak Chikhaoui. On the relationship between the optimal derivative and asymptotic stability. *Afr. Diaspora J. Math. (N.S.)*, 8(2):148–162, 2009.

[12] Tayeb Benouaz, Fatiha Lassouani, Sidi Mohammed Amine Bekkouche, and Martin Bohner. Application of the optimal derivative to the study of a ratio-dependent model describing the evolution of HIV in Canada. *Adv. Dyn. Syst. Appl.*, 6(1):3–12, 2011.

[13] Luitzen E. J. Brouwer. Über Abbildung von Mannigfaltigkeiten. *Math. Ann.*, 71(1):97–115, 1911.

[14] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.