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Newly Defined Conformable Derivatives

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Abstract

Motivated by a proportional-derivative (PD) controller, a more precise definition of a conformable derivative is introduced and explored. Results include basic conformable derivative and integral rules, Taylor's theorem, reduction of order, variation of parameters, complete characterization of solutions for constant coefficient and Cauchy-Euler type conformable equations, Cauchy functions, variation of constants, a self-adjoint equation, and Sturm–Liouville problems.

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1 Introduction

Recently a new local, limit-based definition of a so-called conformable derivative has been formulated [1, 16] via

$$D^\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

or in [14] as

$$D^\alpha f(t) := \lim_{\varepsilon \to 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \quad D^\alpha f(0) = \lim_{t \to 0^+} D^\alpha f(t),$$

provided the limits exist; note that if f is fully differentiable at t, then in either case we have

$$D^\alpha f(t) = t^{1-\alpha} f'(t), \tag{1.1}$$

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where $f'(t) = \lim_{\varepsilon \to 0}[f(t+\varepsilon) - f(t)]/\varepsilon$. A function f is $\alpha-$differentiable at a point $t \geq 0$ if the limits above exist and are finite. Several follow-up papers using at least one of the above conformable definitions include [2–8, 10–13, 19].

The adjective conformable may or may not be appropriate here, since $D^0 f \neq f$, that is to say, letting $\alpha \to 0$ does not result in the identity operator. Moreover, according to (1.1), the variable t must satisfy $t \geq 0$.

With this in mind, in this paper we introduce a new, more precise definition of a conformable derivative of order α for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, where D^0 will be the identity operator, and D^1 will be the classical differential operator.

The conformable derivative was initially referred to as a conformable fractional derivative [1, 14, 16]. However, it lacks some of the agreed upon properties for fractional derivatives [18]. Although the more general definition of the conformable derivative provided below, in Definition 1.1, satisfies some of the properties of a fractional derivative, it is best to consider the conformable derivative in its own right, independent of fractional derivative theory.

Definition 1.1 (Conformable Differential Operator)**.** Let $\alpha \in [0, 1]$. A differential operator D^α is conformable if and only if D^0 is the identity operator and D^1 is the classical differential operator. Specifically, D^α is conformable if and only if for a differentiable function $f = f(t)$,

$$D^0 f(t) = f(t) \quad \text{and} \quad D^1 f(t) = \frac{d}{dt} f(t) = f'(t).$$

Note that under this definition the operator given via (1.1) is not conformable.

Remark 1.2. In control theory, a proportional-derivative (PD) controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and E is the error between the state variable and the process variable; see [17], for example. This is the impetus for the next definition.

Definition 1.3 (A Class of Conformable Derivatives)**.** Let $\alpha \in [0, 1]$, and let the functions $\kappa_0, \kappa_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous such that

$$\begin{aligned} &\lim_{\alpha \to 0^+} \kappa_1(\alpha, t) = 1, && \lim_{\alpha \to 0^+} \kappa_0(\alpha, t) = 0, && \forall\, t \in \mathbb{R}, \\ &\lim_{\alpha \to 1^-} \kappa_1(\alpha, t) = 0, && \lim_{\alpha \to 1^-} \kappa_0(\alpha, t) = 1, && \forall\, t \in \mathbb{R}, \\ &\kappa_1(\alpha, t) \neq 0, \alpha \in [0, 1), && \kappa_0(\alpha, t) \neq 0, \alpha \in (0, 1], && \forall\, t \in \mathbb{R}. \end{aligned} \tag{1.2}$$

Then the following differential operator D^α, defined via

$$D^\alpha f(t) = \kappa_1(\alpha, t) f(t) + \kappa_0(\alpha, t) f'(t) \tag{1.3}$$

is conformable provided the function f is differentiable at t and $f' := \dfrac{d}{dt}f$. Here, κ_1 is a type of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, f is the error, and $u = D^\alpha f$ is the controller output. For example, one could take $\kappa_1 \equiv (1-\alpha)\omega^\alpha$ and $\kappa_0 \equiv \alpha\omega^{1-\alpha}$ for any $\omega \in (0,\infty)$; or, $\kappa_1 = (1-\alpha)|t|^\alpha$ and $\kappa_0 = \alpha|t|^{1-\alpha}$ on $\mathbb{R}\backslash\{0\}$, so that

$$D^\alpha f(t) = (1-\alpha)|t|^\alpha f(t) + \alpha|t|^{1-\alpha}f'(t).$$

A similar class of conformable derivatives could take the form

$$D^\alpha f(t) = \cos(\alpha\pi/2)|t|^\alpha f(t) + \sin(\alpha\pi/2)|t|^{1-\alpha}f'(t).$$

Note that unfortunately $D^\beta D^\alpha \neq D^\alpha D^\beta$ for $\alpha, \beta \in [0,1]$ in general. Also, for $\alpha \in (0,1)$, if we relax the nonzero condition on κ_1 in (1.2) and for $t > 0$ we take $\kappa_1(\alpha,t) \equiv 0$ and $\kappa_0(\alpha,t) = \alpha t^{1-\alpha}$, then for t replaced by $\alpha^{\frac{-1}{1-\alpha}}T$ we have

$$\kappa_0(\alpha,t) = \kappa_0\left(\alpha, \alpha^{\frac{-1}{1-\alpha}}T\right) = \alpha\left(\alpha^{\frac{-1}{1-\alpha}}T\right)^{1-\alpha} = T^{1-\alpha},$$

and we recover (1.1). Thus for $\alpha \in (0,1)$, definition (1.1) is in some sense a special case of (1.3).

Definition 1.4 (Partial Conformable Derivatives)**.** Let $\alpha \in [0,1]$, and let the functions $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous and satisfy (1.2). Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\dfrac{\partial}{\partial t}f(t,s)$ exists for each fixed $s \in \mathbb{R}$, define the partial differential operator D_t^α via

$$D_t^\alpha f(t,s) = \kappa_1(\alpha,t)f(t,s) + \kappa_0(\alpha,t)\frac{\partial}{\partial t}f(t,s). \tag{1.4}$$

Remark 1.5 (Extension to Time Scales). Let $\mathbb{T}$ be a time scale (any nonempty closed subset of real numbers). Then (1.3) can be extended to $\mathbb{T}$ via

$$D^\alpha f(t) = \kappa_1(\alpha,t)f(t) + \kappa_0(\alpha,t)f^\Delta(t)$$

for Δ-differentiable functions f on $\mathbb{T}$, where f^Δ is the delta derivative of f. This includes and generalizes the conformable time scales derivative introduced in [9].

Definition 1.6 (Conformable Exponential Function)**.** Let $\alpha \in (0,1]$, the points $s, t \in \mathbb{R}$ with $s \leq t$, and let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous and satisfy (1.2), with p/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. Then the exponential function with respect to D^α in (1.3) is defined to be

$$e_p(t,s) := e^{\int_s^t \frac{p(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau}, \quad e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau}. \tag{1.5}$$

Using (1.3) and (1.5) we have the following basic results.

Lemma 1.7 (Basic Derivatives)**.** *Let the conformable differential operator D^α be given as in* (1.3)*, where $\alpha \in [0,1]$. Let the function $p : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.2)*, with p/κ_0 and κ_1/κ_0 Riemann integrable on $[s,t]$. Assume the functions f and g are differentiable as needed. Then*

(i) $D^\alpha[af+bg] = aD^\alpha[f] + bD^\alpha[g]$ *for all* $a,b \in \mathbb{R}$*;*

(ii) $D^\alpha c = c\kappa_1(\alpha,\cdot)$ *for all constants* $c \in \mathbb{R}$*;*

(iii) $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f] - fg\kappa_1(\alpha,\cdot)$*;*

(iv) $D^\alpha[f/g] = \dfrac{gD^\alpha[f] - fD^\alpha[g]}{g^2} + \dfrac{f}{g}\kappa_1(\alpha,\cdot)$*;*

(v) for $\alpha \in (0,1]$ *and fixed* $s \in \mathbb{R}$*, the exponential function satisfies*

$$D_t^\alpha\left[e_p(t,s)\right] = p(t)e_p(t,s) \tag{1.6}$$

for $e_p(t,s)$ *given in* (1.5)*;*

(vi) for $\alpha \in (0,1]$ *and for the exponential function* e_0 *given in* (1.5)*, we have*

$$D^\alpha\left[\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\right] = f(t). \tag{1.7}$$

Proof. Items (i) and (ii) follow easily from (1.3). For (iii), we use (1.3) to get that (suppressing the variable)

$$\begin{aligned} D^\alpha[fg] &= \kappa_0(fg' + f'g) + \kappa_1 fg \\ &= (f\kappa_0 g' + f\kappa_1 g) + (g\kappa_0 f' + g\kappa_1 f) - fg\kappa_1 = fD^\alpha g + gD^\alpha f - fg\kappa_1. \end{aligned}$$

The proof of (iv) is similar and is omitted. To prove (v), apply the partial derivative (1.4) to e_p from (1.5) to obtain

$$\begin{aligned} D_t^\alpha e_p(t,s) &= \kappa_0(\alpha,t)\left(\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\right)e_p(t,s) + \kappa_1(\alpha,t)e_p(t,s) \\ &= \left(p(t)-\kappa_1(\alpha,t)\right)e_p(t,s) + \kappa_1(\alpha,t)e_p(t,s) = p(t)e_p(t,s). \end{aligned}$$

Finally for (vi), using (1.3) and (1.5) again we have

$$\begin{aligned} D^\alpha\left[\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\right] &= \kappa_0(\alpha,t)\cdot\frac{d}{dt}\left(\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\right) \\ &\quad +\kappa_1(\alpha,t)\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds \end{aligned}$$

$$
\begin{aligned}
&= \kappa_0(\alpha,t)\left(\frac{-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds + \frac{f(t)e_0(t,t)}{\kappa_0(\alpha,t)}\right)\\
&\quad +\kappa_1(\alpha,t)\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\\
&= -\kappa_1(\alpha,t)\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds + f(t)\\
&\quad +\kappa_1(\alpha,t)\int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\\
&= f(t),
\end{aligned}
$$

and the proof is complete. □

Definition 1.8 (Integrals)**.** Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. In light of (1.5) and Lemma 1.7 (v) & (vi), define the antiderivative via

$$\int D^\alpha f(t) d_\alpha t = f(t) + c e_0(t,t_0), \quad c \in \mathbb{R}.$$

Similarly, define the integral of f over $[a,b]$ as

$$\int_a^t f(s)e_0(t,s)d_\alpha s := \int_a^t \frac{f(s)e_0(t,s)}{\kappa_0(\alpha,s)}ds, \quad d_\alpha s := \frac{1}{\kappa_0(\alpha,s)}ds; \tag{1.8}$$

recall that

$$e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau} = e^{-\int_s^t \kappa_1(\alpha,\tau)d_\alpha\tau}$$

from (1.5).

Lemma 1.9 (Basic Integrals)**.** *Let the conformable differential operator D^α be given as in* (1.3)*, the integral be given as in* (1.8) *with $\alpha \in (0,1]$. Let the functions κ_0, κ_1 be continuous and satisfy* (1.2)*, and let f and g be differentiable as needed. Then*

(i) the derivative of the definite integral of f is given by

$$D^\alpha\left[\int_a^t f(s)e_0(t,s)d_\alpha s\right] = f(t);$$

(ii) the definite integral of the derivative of f is given by

$$\int_a^t D^\alpha[f(s)]e_0(t,s)d_\alpha s = f(s)e_0(t,s)\big|_{s=a}^t := f(t) - f(a)e_0(t,a);$$

(iii) an integration by parts formula is given by

$$
\begin{aligned}
\int_a^b f(t)D^\alpha[g(t)]e_0(b,t)d_\alpha t &= f(t)g(t)e_0(b,t)\big|_{t=a}^b\\
&\quad -\int_a^b g(t)\left(D^\alpha[f(t)]-\kappa_1(\alpha,t)f(t)\right)e_0(b,t)d_\alpha t;
\end{aligned}
$$

(iv) a version of the Leibniz rule for differentiation of an integral is given by

$$D^\alpha\left[\int_a^t f(t,s)e_0(t,s)d_\alpha s\right] = \int_a^t \left(D_t^\alpha[f(t,s)] - \kappa_1(\alpha,t)f(t,s)\right)e_0(t,s)d_\alpha s + f(t,t),$$

using (1.4)*; or, if* e_0 *is absent,*

$$D^\alpha\left[\int_a^t f(t,s)d_\alpha s\right] = f(t,t) + \int_a^t D_t^\alpha[f(t,s)]d_\alpha s.$$

Proof. The proof of (i) follows directly from (1.7) and (1.8). Using Lemma 1.7 (ii) with $c = 1$, (ii) here is a special case of (iii). To prove (iii), use Lemma 1.7 (iii) and the definition of the integral in (1.8). For the second expression in (iii), we have used (1.3), (1.8), and $\alpha \neq 0$ to see that

$$f'(t)dt = \frac{D^\alpha f(t) - \kappa_1(\alpha,t)f(t)}{\kappa_0(\alpha,t)}dt = \left(D^\alpha f(t) - \kappa_1(\alpha,t)f(t)\right)d_\alpha t.$$

To prove (iv), we use the ($\alpha = 1$) Leibniz rule to get

$$\begin{aligned} D^\alpha\left[\int_a^t f(t,s)e_0(t,s)d_\alpha s\right] &= D^\alpha\left[\int_a^t \frac{f(t,s)e_0(t,s)}{\kappa_0(\alpha,s)}ds\right] \\ &= \kappa_0(\alpha,t)\frac{d}{dt}\int_a^t \frac{f(t,s)e_0(t,s)}{\kappa_0(\alpha,s)}ds \\ &\quad + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)e_0(t,s)}{\kappa_0(\alpha,s)}ds \\ &= \kappa_0(\alpha,t)\int_a^t \frac{e_0(t,s)}{\kappa_0(\alpha,s)}\left(\frac{-\kappa_1(\alpha,t)f(t,s)}{\kappa_0(\alpha,t)}\right. \\ &\quad \left. + \frac{\partial}{\partial t}f(t,s)\right)ds \\ &\quad + f(t,t) + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)e_0(t,s)}{\kappa_0(\alpha,s)}ds \\ &= \int_a^t \left(D_t^\alpha[f(t,s)] - \kappa_1(\alpha,t)f(t,s)\right)e_0(t,s)d_\alpha s \\ &\quad + f(t,t). \end{aligned}$$

For the second expression in (iv), if $e_0(t,s)$ is absent from the integral expression, then

$$\begin{aligned} D^\alpha\left[\int_a^t f(t,s)d_\alpha s\right] &= D^\alpha\left[\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}ds\right] \\ &= \kappa_0(\alpha,t)\frac{d}{dt}\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}ds + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)}ds \end{aligned}$$

$$
\begin{aligned}
&= \kappa_0(\alpha,t)\left[\frac{f(t,t)}{\kappa_0(\alpha,t)} + \int_a^t \frac{\frac{\partial}{\partial t} f(t,s)}{\kappa_0(\alpha,s)} ds\right] \\
&\quad + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)}{\kappa_0(\alpha,s)} ds \\
&= f(t,t) + \int_a^t D_t^\alpha[f(t,s)] d_\alpha s,
\end{aligned}
$$

and the proof is complete. □

A useful result for solving first-order conformable differential equations is given in the following lemma.

Lemma 1.10 (Variation of Constants). *Assume* κ_0, κ_1 *satisfy* (1.2)*. Let* $f, p : [t_0, \infty) \to \mathbb{R}$ *be continuous, let* e_p *be as in* (1.5)*, and let* $u_0 \in \mathbb{R}$*. Then the unique solution of the initial value problem*

$$D^\alpha u(t) - p(t)u(t) = f(t), \quad u(t_0) = u_0,$$

is given by

$$u(t) = u_0 e_p(t, t_0) + \int_{t_0}^t e_p(t,s) f(s) d_\alpha s, \quad t \in [t_0, \infty). \tag{1.9}$$

Proof. Let u be given by (1.9). Using Lemma 1.7 (v) and Lemma 1.9 (iv),

$$
\begin{aligned}
D^\alpha u(t) &= u_0 p(t) e_p(t,t_0) + e_p(t,t) f(t) + \int_{t_0}^t p(t) e_p(t,s) f(s) d_\alpha s \\
&= p(t)u(t) + f(t),
\end{aligned}
$$

which completes the proof of the lemma. □

2 Taylor Series

Now that we have integration defined, we will introduce core functions that will serve the role that polynomials do in Taylor series expansions for the regular derivative ($\alpha = 1$). Let the functions $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous such that the conditions in (1.2) are satisfied. When $\alpha = 1$ and $n \in \mathbb{N}_0$, the polynomials are given by $h_n(t,s) = \frac{1}{n!}(t-s)^n$. To generalize this to the present context, define recursively the functions $h_n : \mathbb{R}^2 \to \mathbb{R}$, $n \in \mathbb{N}_0$ via

$$h_0(t,s) \equiv 1 \quad \text{for all} \quad t, s \in \mathbb{R} \tag{2.1}$$

and

$$h_n(t,s) = \int_s^t h_{n-1}(\tau, s) d_\alpha \tau, \quad n \in \mathbb{N}, \quad \text{for all} \quad t, s \in \mathbb{R}. \tag{2.2}$$

By Lemma 1.7 (ii) and Lemma 1.9 (iv), we have the key relationship

$$D_t^\alpha h_n(t,s) = h_{n-1}(t,s) + \kappa_1(\alpha,t)h_n(t,s). \tag{2.3}$$

Before we present and prove Taylor's formula for the conformable derivative (1.3), we need the following preliminary result.

Lemma 2.1. *Let $n \in \mathbb{N}$. If f is n times differentiable and the functions p_k, $0 \le k \le n-1$, are differentiable at some $t \in \mathbb{R}$ with*

$$D^\alpha p_{k+1}(t) = p_k(t) + \kappa_1(\alpha,t)p_{k+1}(t) \quad \textit{for all} \quad 0 \le k \le n-2, \tag{2.4}$$

then we have

$$D^\alpha \left[\sum_{k=0}^{n-1}(-1)^k p_k (D^\alpha)^k f\right] = (-1)^{n-1}p_{n-1}(D^\alpha)^n f + \left(D^\alpha p_0 - \kappa_1(\alpha,\cdot)p_0\right) f$$

at t.

Proof. Using Lemma 1.7 (i) & (iii) and (2.4), we find that

$$\begin{aligned}
D^\alpha \left[\sum_{k=0}^{n-1}(-1)^k p_k (D^\alpha)^k f\right] &= \sum_{k=0}^{n-1}(-1)^k D^\alpha \left[p_k (D^\alpha)^k f\right] \\
&= \sum_{k=0}^{n-1}(-1)^k \left\{p_k (D^\alpha)^{k+1} f + \left(D^\alpha p_k - \kappa_1(\alpha,\cdot)p_k\right)(D^\alpha)^k f\right\} \\
&= \sum_{k=0}^{n-2}(-1)^k p_k (D^\alpha)^{k+1} f + (-1)^{n-1}p_{n-1}(D^\alpha)^n f \\
&\quad + \sum_{k=1}^{n-1}(-1)^k \left(D^\alpha p_k - \kappa_1(\alpha,\cdot)p_k\right)(D^\alpha)^k f + \left(D^\alpha p_0 - \kappa_1(\alpha,\cdot)p_0\right) f \\
&= \sum_{k=0}^{n-2}(-1)^k p_k (D^\alpha)^{k+1} f + (-1)^{n-1}p_{n-1}(D^\alpha)^n f \\
&\quad + \sum_{k=0}^{n-2}(-1)^{k+1} \left(D^\alpha p_{k+1} - \kappa_1(\alpha,\cdot)p_{k+1}\right)(D^\alpha)^{k+1} f + \left(D^\alpha p_0 - \kappa_1(\alpha,\cdot)p_0\right) f \\
&= (-1)^{n-1}p_{n-1}(D^\alpha)^n f + \left(D^\alpha p_0 - \kappa_1(\alpha,\cdot)p_0\right) f
\end{aligned}$$

holds at t. This proves the lemma. □

Theorem 2.2 (Taylor's Formula)**.** *Let $n \in \mathbb{N}$, and suppose f is n times differentiable on $[t_0,\infty)$. Let $t, s \in [t_0,\infty)$, and define the functions h_k by* (2.1) *and* (2.2)*, i.e.,*

$$h_0(t,s) \equiv 1 \quad \textit{and} \quad h_{k+1}(t,s) = \int_s^t h_k(\tau,s)d_\alpha\tau \textit{ for } k \in \mathbb{N}_0.$$

Then we have

$$\begin{aligned} f(t) &= e_0(t,s)\sum_{k=0}^{n-1}(-1)^k h_k(s,t)(D^\alpha)^k f(s) \\ &\quad +(-1)^{n-1}\int_s^t h_{n-1}(\tau,t)(D^\alpha)^n f(\tau)e_0(t,\tau)d_\alpha\tau \end{aligned}$$

for $t \in [t_0,\infty)$.

Proof. By (2.3) and Lemma 2.1 we have

$$D^\alpha\left[\sum_{k=0}^{n-1}(-1)^k h_k(\cdot,t)(D^\alpha)^k f\right](\tau) = (-1)^{n-1}h_{n-1}(\tau,t)(D^\alpha)^n f(\tau)$$

for all $\tau \in [t_0,\infty)$. Integrating the above equation from s to t we obtain

$$\begin{aligned} &(-1)^{n-1}\int_s^t h_{n-1}(\tau,t)(D^\alpha)^n f(\tau)e_0(t,\tau)d_\alpha\tau \\ &= \int_s^t D^\alpha\left[\sum_{k=0}^{n-1}(-1)^k h_k(\cdot,t)(D^\alpha)^k f\right](\tau)e_0(t,\tau)d_\alpha\tau \\ &= \sum_{k=0}^{n-1}(-1)^k h_k(\tau,t)(D^\alpha)^k f(\tau)e_0(t,\tau)\Big|_{\tau=s}^{t} \\ &= f(t) - \sum_{k=0}^{n-1}(-1)^k h_k(s,t)(D^\alpha)^k f(s)e_0(t,s), \end{aligned}$$

where we used Lemma 1.9 (ii). □

In the next few examples, we explore these new functions $h_n(t,s)$ given in (2.2).

Example 2.3. For $\alpha \in (0,1]$, let $\omega_0,\omega_1 \in (0,\infty)$, let κ_1 satisfy (1.2), and take

$$\kappa_0(\alpha,t) \equiv \alpha\omega_0^{1-\alpha}.$$

By (1.8),

$$d_\alpha\tau = \frac{1}{\kappa_0(\alpha,\tau)}d\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}d\tau.$$

Letting $h_0(t,s) \equiv 1$ as in (2.1), we calculate h_1 via (2.2) to get

$$h_1(t,s) = \int_s^t h_0(\tau,s)d_\alpha\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}\int_s^t 1d\tau = \frac{t-s}{\alpha\omega_0^{1-\alpha}};$$

additionally,

$$h_2(t,s) = \int_s^t h_1(\tau,s)d_\alpha\tau = \frac{1}{2!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^2.$$

In general we have that

$$h_n(t,s) = \frac{1}{n!}\left(\frac{t-s}{\alpha\omega_0^{1-\alpha}}\right)^n.$$

Note that at $\alpha = 1$ we have

$$h_n(t,s) = \frac{1}{n!}(t-s)^n$$

as expected. □

Example 2.4. For $\alpha \in (0,1]$, let $\omega_0, \omega_1 \in (0,\infty)$, let κ_1 satisfy (1.2), and this time take

$$\kappa_0(\alpha,t) = \alpha(\omega_0 t)^{1-\alpha}, \quad t \in [0,\infty).$$

By (1.8),

$$d_\alpha\tau = \frac{\tau^{\alpha-1}}{\alpha\omega_0^{1-\alpha}}d\tau.$$

Again starting with $h_0(t,s) \equiv 1$, we see that

$$h_1(t,s) = \int_s^t h_0(\tau,s)d_\alpha\tau = \frac{1}{\alpha\omega_0^{1-\alpha}}\int_s^t \tau^{\alpha-1}d\tau = \frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}},$$

and

$$h_2(t,s) = \int_s^t h_1(\tau,s)d_\alpha\tau = \frac{1}{2!}\left(\frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}}\right)^2.$$

Continuing, we find that

$$h_n(t,s) = \frac{1}{n!}\left(\frac{t^\alpha - s^\alpha}{\alpha^2\omega_0^{1-\alpha}}\right)^n,$$

which is just $\frac{1}{n!}(t-s)^n$ at $\alpha = 1$. □

Example 2.5. Let

$$\kappa_0(\alpha,t) = t^{1-\alpha}\cos^2\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right) \quad \text{and} \quad \kappa_1(\alpha,t) = \sin^2\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right).$$

One can readily verify that these functions satisfy (1.2). Of course $h_0 \equiv 1$, so

$$h_1(t,s) = \int_s^t \frac{\tau^{\alpha-1}d\tau}{\cos^2\left(\frac{\pi}{2}(1-\alpha)\tau^\alpha\right)}.$$

The substitution $u = \tau^\alpha$, $du = \alpha\tau^{\alpha-1}d\tau$ gives

$$h_1(t,s) = \frac{1}{\alpha}\int_{s^\alpha}^{t^\alpha} \frac{du}{\cos^2\left(\frac{\pi}{2}(1-\alpha)u\right)}$$

$$= \frac{\tan\left(\frac{\pi}{2}(1-\alpha)t^{\alpha}\right) - \tan\left(\frac{\pi}{2}(1-\alpha)s^{\alpha}\right)}{\alpha(1-\alpha)\frac{\pi}{2}}.$$

One verifies the recovery of the regular polynomial by (for convenience) letting $\chi \equiv (1-\alpha)\dfrac{\pi}{2}$ and taking the limit as $\chi \to 0^+$ (meaning $\alpha \to 1$). First, series expansion gives

$$h_1(t,s) = \frac{\tan\chi t^{\alpha} - \tan\chi s^{\alpha}}{\alpha\chi} = \frac{\chi t^{\alpha} - \chi s^{\alpha}}{\alpha\chi} + \mathcal{O}\left[\chi^2\right],$$

and then

$$\lim_{\chi\to 0^+(\alpha\to 1)} h_1(t,s) = t - s.$$

The $h_n(t,s)$ are a bit more complicated than in the previous examples, but nevertheless are amenable to solution. First,

$$\begin{aligned} h_2(t,s) &= \int_s^t h_1(\tau,s)d_{\alpha}\tau \\ &= \frac{1}{\alpha\chi}\int_s^t \frac{\left(\tan\chi\tau^{\alpha} - \tan\chi s^{\alpha}\right)\tau^{\alpha-1}d\tau}{\cos^2\chi\tau^{\alpha}}. \end{aligned}$$

Again, the substitution $u = \tau^{\alpha}$, $du = \alpha\tau^{\alpha-1}d\tau$ gives

$$\begin{aligned} h_2(t,s) &= \frac{1}{2\alpha^2\chi}\int_{s^{\alpha}}^{t^{\alpha}} \frac{\tan\chi u - \tan\chi s^{\alpha}du}{\cos^2\chi u} \\ &= \frac{1}{2\alpha^2\chi^2}\left(\sec^2\chi t^{\alpha} - \sec^2\chi s^{\alpha} - 2\tan\chi t^{\alpha}\tan\chi s^{\alpha} + 2\left(\tan\chi s^{\alpha}\right)^2\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \lim_{\chi\to 0^+(\alpha\to 1)} h_2(t,s) &= \lim_{\chi\to 0^+(\alpha\to 1)} \frac{1}{2\alpha^2\chi^2}\left(1 - 1 + (\chi t^{\alpha})^2 - (\chi s^{\alpha})^2 - 2\chi t^{\alpha}\chi s^{\alpha}\right. \\ &\quad \left. + 2\left(\chi s^{\alpha}\right)^2 + \mathcal{O}\left[\chi^4\right]\right) \\ &= \lim_{\chi\to 0^+(\alpha\to 1)} \frac{1}{2\alpha^2\chi^2}\left((\chi t^{\alpha})^2 - 2\chi t^{\alpha}\chi s^{\alpha} + (\chi s^{\alpha})^2 + \mathcal{O}\left[\chi^4\right]\right) \\ &= \lim_{\chi\to 0^+(\alpha\to 1)} \frac{1}{2\alpha^2}\left((t^{\alpha} - s^{\alpha})^2 + \mathcal{O}\left[\chi^2\right]\right) \\ &= \frac{1}{2}\left(t-s\right)^2. \end{aligned}$$

One can consider the exponential function for this example as well. Here,

$$e_{\lambda}(t,s) = e^{\int_s^t \frac{\lambda - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau}.$$

The integral can be evaluated to

$$\begin{aligned}\int_s^t \frac{\lambda-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}d\tau &= \int_s^t \frac{\lambda-\sin^2\left(\frac{\pi}{2}(1-\alpha)\tau^\alpha\right)}{\tau^{1-\alpha}\cos^2\left(\frac{\pi}{2}(1-\alpha)\tau^\alpha\right)}d\tau \\ &= \frac{(\lambda-1)\tan\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}}+\frac{t^\alpha}{\alpha}+C,\end{aligned}$$

where the constant C is given by

$$C=-\frac{(\lambda-1)\tan\left(\frac{\pi}{2}(1-\alpha)s^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}}-\frac{s^\alpha}{\alpha}.$$

Thus,

$$e_\lambda(t,s)=Ae^{\frac{(\lambda-1)\tan\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}}+\frac{t^\alpha}{\alpha}},$$

where $A=e^C$. Note the special case,

$$e_1(t,s)=Ae^{\frac{t^\alpha}{\alpha}}.$$

Also note that again with $\chi\equiv(1-\alpha)\frac{\pi}{2}$ we have

$$\lim_{\chi\to 0^+(\alpha\to 1)} e_\lambda(t,s)=\lim_{\chi\to 0^+(\alpha\to 1)} Ae^{(\lambda-1)\frac{\chi t^\alpha}{\alpha\chi}+\frac{t^\alpha}{\alpha}+\mathcal{O}[\chi^2]}=Ae^{\lambda t}.$$

3 Second-Order Linear Conformable Derivatives

An important equation in mathematics, mathematical physics, mathematical biology, physical chemistry, and engineering is the second–order linear homogeneous ordinary differential equation with constant coefficients given by

$$ay''(t)+by'(t)+cy(t)=0,\quad t\in\mathbb{R}.$$

In a spring-mass system, one thinks of a as mass, b as the damping coefficient, and c as the spring constant. Using (1.3), in this section we will explore the analogous second-order linear homogeneous conformable differential equation with constant coefficients

$$aD^\alpha D^\alpha y(t)+bD^\alpha y(t)+cy(t)=0,\quad t\in[t_0,\infty).$$

In addition, we will also analyze the related Cauchy-Euler type conformable equation

$$atD^\alpha\left[tD^\alpha y(t)\right]+btD^\alpha y(t)+cy(t)=0,\quad t\in[t_0,\infty),\quad t_0>0.$$

Theorem 3.1 (Constant Coefficient CDEs)**.** *Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.2)*, and let D^α be as given in* (1.3)*. Let $a, b, c \in \mathbb{R}$ be constants and $\alpha \in (0,1]$. From* (1.5) *recall that*

$$e_\lambda(t,t_0) = e^{\int_{t_0}^{t} \frac{\lambda-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau} = e^{\int_{t_0}^{t} (\lambda-\kappa_1(\alpha,\tau)) d_\alpha \tau}$$

for constant λ. Then the constant coefficient homogeneous conformable differential equation

$$aD^\alpha D^\alpha y(t) + bD^\alpha y(t) + cy(t) = 0, \quad t \in [t_0, \infty), \tag{3.1}$$

has the associated auxiliary equation

$$a\lambda^2 + b\lambda + c = 0, \tag{3.2}$$

and the general solution to (3.1) *is given by one of the following for constants $c_1, c_2 \in \mathbb{R}$:*

(i) if $\lambda_1, \lambda_2 \in \mathbb{R}$ are distinct roots of (3.2)*, then*

$$y(t) = c_1 e_{\lambda_1}(t,t_0) + c_2 e_{\lambda_2}(t,t_0);$$

(ii) if $\lambda = \dfrac{-b}{2a}$ is a repeated root of (3.2)*, then*

$$y(t) = c_1 e_\lambda(t,t_0) + c_2 e_\lambda(t,t_0) \int_{t_0}^{t} 1 d_\alpha s;$$

(iii) if $\lambda = \zeta \pm i\beta$ is a complex root of (3.2)*, then*

$$y(t) = c_1 e_\zeta(t,t_0) \cos\left(\int_{t_0}^{t} \beta d_\alpha s \right) + c_2 e_\zeta(t,t_0) \sin\left(\int_{t_0}^{t} \beta d_\alpha s \right).$$

Proof. Begin with the trial solution

$$y = y(t) = e_\lambda(t,t_0) = e^{\int_{t_0}^{t} \frac{\lambda-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau},$$

where λ is a complex constant to be determined. Plugging y into (3.1), this leads via (1.6) to the same auxiliary equation as in the classical case of $\alpha = 1$, namely (3.2). Thus there are three cases. If there are two real and distinct roots to the auxiliary equation, the result is a linear combination of exponentials for the general solution as given in (i) above.

Suppose $\lambda \in \mathbb{R}$ is a root of (3.2) of multiplicity two, namely $\lambda = \dfrac{-b}{2a}$. Then

$$y_1(t) = e_\lambda(t,t_0) = e_{\frac{-b}{2a}}(t,t_0)$$

is one solution of (3.1). When $\alpha = 1$, we know that $y_2 = ty_1$ is a second linearly independent solution, so we try

$$y_2(t) = e_\lambda(t,t_0)h_1(t,t_0) = e_\lambda(t,t_0)\int_{t_0}^{t} 1 d_\alpha s.$$

Thus, using (1.3) or (2.3), we have

$$D^\alpha y_2(t) = e_\lambda(t,t_0)\left(1+\lambda\int_{t_0}^{t} 1 d_\alpha s\right),\ D^\alpha D^\alpha y_2(t) = e_\lambda(t,t_0)\left(2\lambda+\lambda^2\int_{t_0}^{t} 1 d_\alpha s\right);$$

checking

$$aD^\alpha D^\alpha y_2(t) + bD^\alpha y_2(t) + cy_2(t)$$

with $\lambda = -b/(2a)$, we see that

$$\begin{aligned} ae_\lambda(t,t_0)\left(2\lambda+\lambda^2\int_{t_0}^{t} 1 d_\alpha s\right) + be_\lambda(t,t_0)\left(1+\lambda\int_{t_0}^{t} 1 d_\alpha s\right) + ce_\lambda(t,t_0)\int_{t_0}^{t} 1 d_\alpha s \\ = e_\lambda(t,t_0)\left(a\lambda^2+b\lambda+c\right)\int_{t_0}^{t} 1 d_\alpha s = 0 \end{aligned}$$

as λ is a zero of the auxiliary polynomial.

Finally, assume the roots of (3.2) are complex numbers of the form $\lambda = \zeta \pm i\beta$. Then a complex-valued solution of (3.1) is

$$y(t) = e_{(\zeta+i\beta)}(t,t_0) = e_\zeta(t,t_0)\left(\cos\left(\int_{t_0}^{t}\beta d_\alpha s\right) + i\sin\left(\int_{t_0}^{t}\beta d_\alpha s\right)\right)$$

by Euler's formula and (1.5); the real and imaginary parts of this expression are linearly independent solutions of (3.1). □

The next theorem supplies the general solutions for second-order linear homogeneous Cauchy-Euler conformable differential equations.

Theorem 3.2 (Cauchy-Euler CDEs). *Let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.2)*, and let D^α be as given in* (1.3)*. Let $a, b, c \in \mathbb{R}$ be constants and $\alpha \in (0,1]$. Then the homogeneous Cauchy-Euler type conformable differential equation*

$$atD^\alpha\left[tD^\alpha y(t)\right] + btD^\alpha y(t) + cy(t) = 0, \quad t\in[t_0,\infty), \quad t_0>0, \tag{3.3}$$

has the associated auxiliary equation (3.2)*, and the general solution to* (3.3) *is given by one of the following for constants $c_1, c_2 \in \mathbb{R}$:*

(i) $y(t) = c_1 e_{\lambda_1/t}(t,t_0) + c_2 e_{\lambda_2/t}(t,t_0)$*, where $\lambda_1, \lambda_2 \in \mathbb{R}$ are distinct roots of* (3.2)*;*

(ii) $y(t) = c_1 e_{\lambda/t}(t,t_0) + c_2 e_{\lambda/t}(t,t_0)\int_{t_0}^{t} s^{-1} d_\alpha s$*, where $\lambda = \dfrac{-b}{2a}$ is a repeated root of* (3.2)*;*

(iii) $y(t) = c_1 e_{\zeta/t}(t,t_0)\cos\left(\beta\int_{t_0}^{t} s^{-1}d_\alpha s\right) + c_2 e_{\zeta/t}(t,t_0)\sin\left(\beta\int_{t_0}^{t} s^{-1}d_\alpha s\right)$*, where* $\lambda = \zeta \pm i\beta$ *is a complex root of* (3.2).

Proof. In this case begin with the trial solution

$$y(t) = e_{\lambda/t}(t,t_0),$$

where λ is a complex constant to be determined, and the exponential function is given as in (1.5). Plugging this y into (3.3) leads to the auxiliary equation (3.2), and thus three cases again. If there are two real and distinct roots to the auxiliary equation, the result is clear and we have (i).

Suppose $\lambda \in \mathbb{R}$ is a root of (3.2) of multiplicity two, namely $\lambda = \dfrac{-b}{2a}$. Then $y_1(t) = e_{\lambda/t}(t,t_0)$ is one solution of (3.3). Setting

$$y_2(t) = y_1(t)\int_{t_0}^{t} s^{-1}d_\alpha s,$$

we see that

$$\begin{aligned} atD^\alpha\left[tD^\alpha y_2(t)\right] + btD^\alpha y_2(t) + cy_2(t) &= 2atD^\alpha y_1(t) + atD^\alpha[tD^\alpha y_1(t)]\int_{t_0}^{t} s^{-1}d_\alpha s \\ &\quad +by_1(t) + (btD^\alpha y_1(t))\int_{t_0}^{t} s^{-1}d_\alpha s \\ &\quad +cy_1(t)\int_{t_0}^{t} s^{-1}d_\alpha s \\ &= 0 \end{aligned}$$

and we have (ii), as y_1 is a solution, and $\lambda = -b/(2a)$.

Finally, assume the roots of (3.2) are complex numbers of the form $\lambda = \zeta \pm i\beta$. Then one complex-valued solution of (3.3) is

$$y(t) = e_{(\zeta+i\beta)/t}(t,t_0) = e_{\zeta/t}(t,t_0)\left(\cos\left(\beta\int_{t_0}^{t} s^{-1}d_\alpha s\right) + i\sin\left(\beta\int_{t_0}^{t} s^{-1}d_\alpha s\right)\right);$$

once again the real and imaginary parts of this expression are linearly independent solutions of (3.3). $\square$

4 Self-Adjoint Conformable Equations

Let $\alpha \in [0,1]$, and let D^α be as in (1.3). In this section we are concerned with the formally self-adjoint equation with two iterated conformable derivatives

$$Lx = 0, \quad \text{where} \quad Lx(t) = D^\alpha\left[p\left(D^\alpha x - \kappa_1(\alpha,\cdot)x\right)\right](t) + q(t)x(t). \tag{4.1}$$

Throughout we assume that p, q are continuous on $[t_0, \infty)$ and

$$p(t) \neq 0 \quad \text{for all} \quad t \in [t_0, \infty).$$

Define the set $\mathbb{D}$ to be the set of all functions $x : [t_0, \infty) \to \mathbb{R}$ such that $D^\alpha x : [t_0, \infty) \to \mathbb{R}$ is continuous and such that $D^\alpha [p (D^\alpha x - \kappa_1(\alpha, \cdot)x)] : [t_0, \infty) \to \mathbb{R}$ is continuous. A function $x \in \mathbb{D}$ is then said to be a solution of (4.1) provided $Lx(t) = 0$ holds for all $t \in [t_0, \infty)$. Recall that κ_0, κ_1 satisfy (1.2), D^α is given in (1.3), and the integral is defined in (1.8).

We next state a theorem concerning the existence-uniqueness of solutions of initial value problems for the inhomogeneous self-adjoint equation $Lx = f(t)$.

Theorem 4.1. *Assume κ_0, κ_1 satisfy* (1.2). *Let $\alpha \in (0, 1]$, and let D^α be as in* (1.3). *Assume p, q, f is continuous on $[t_0, \infty)$ with $p(t) \neq 0$, and suppose $x_0, x_1 \in \mathbb{R}$ are given constants. Then the initial value problem*

$$Lx = f(t), \quad x(t_0) = x_0, \quad D^\alpha x(t_0) = x_1$$

has a unique solution that exists on $[t_0, \infty)$.

Proof. We will write $Lx = f$ as an equivalent vector equation, and then invoke the standard ($\alpha = 1$) result to complete the argument. Let x be a solution of $Lx = f$, and set

$$y(t) := p(t) \left(D^\alpha x(t) - \kappa_1(\alpha, t)x(t) \right),$$

so that

$$D^\alpha x(t) = \kappa_1(\alpha, t)x(t) + \frac{1}{p(t)} y(t).$$

Using the fact that x is a solution of $Lx = f$ for L defined in (4.1), we have

$$D^\alpha y(t) = -q(t)x(t) + f(t).$$

Therefore, if we set the vector

$$z(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

then z is a solution of the vector equation

$$D^\alpha z(t) = A(t)z(t) + b(t), \quad A(t) = \begin{bmatrix} \kappa_1(\alpha, t) & 1/p(t) \\ -q(t) & 0 \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

By the definition of D^α in (1.3), we thus have

$$z'(t) = \begin{bmatrix} 0 & \dfrac{1}{\kappa_0(\alpha, t)p(t)} \\ \dfrac{-q(t)}{\kappa_0(\alpha, t)} & \dfrac{-\kappa_1(\alpha, t)}{\kappa_0(\alpha, t)} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \dfrac{f(t)}{\kappa_0(\alpha, t)} \end{bmatrix},$$

and the result follows from the standard ($\alpha = 1$) proof, since all the functions here are continuous; see, for example, [15, Theorem 5.4]. $\square$

Example 4.2. Assume κ_0, κ_1 satisfy (1.2), such that κ_1 is differentiable on $[0, \infty)$. Let $\alpha \in (0, 1]$, and let $x_0, x_1 \in \mathbb{R}$ be given constants. If

$$p(t) := e_{2\kappa_1}(t, 0) = e_0(0, t), \quad q(t) := (1 + D^\alpha \kappa_1(\alpha, t))\, p(t), \quad t \in [0, \infty),$$

then one can check that (4.1) reduces to $D^\alpha D^\alpha x + x = 0$. The unique solution to the initial value problem

$$D^\alpha D^\alpha x + x = 0, \quad x(0) = x_0, \quad D^\alpha x(0) = x_1$$

is given by

$$x(t) = x_0 e_0(t, 0) \cos\left(\int_0^t 1 d_\alpha s\right) + x_1 e_0(t, 0) \sin\left(\int_0^t 1 d_\alpha s\right),$$

which follows from Theorem 3.1. If $\omega_0, \omega_1 \in (0, \infty)$ and

$$\kappa_0(\alpha, t) \equiv \alpha\omega_0^{1-\alpha}, \qquad \kappa_1(\alpha, t) \equiv (1 - \alpha)\omega_1^\alpha,$$

then by (1.5) and (1.8) we have

$$e_0(t, 0) = e^{\left(1-\frac{1}{\alpha}\right)\omega_0^{\alpha-1}\omega_1^\alpha t}, \quad d_\alpha s = \frac{1}{\kappa_0(\alpha, s)} ds = \frac{1}{\alpha\omega_0^{1-\alpha}} ds, \quad \int_0^t 1 d_\alpha s = \frac{t}{\alpha\omega_0^{1-\alpha}}.$$

Thus the solution in this example is

$$x(t) = x_0 e^{\left(1-\frac{1}{\alpha}\right)\omega_0^{\alpha-1}\omega_1^\alpha t} \cos\left(\frac{t}{\alpha\omega_0^{1-\alpha}}\right) + x_1 e^{\left(1-\frac{1}{\alpha}\right)\omega_0^{\alpha-1}\omega_1^\alpha t} \sin\left(\frac{t}{\alpha\omega_0^{1-\alpha}}\right).$$

It is easy to see that as $\alpha \to 1$, $x(t) \to x_0 \cos t + x_1 \sin t$, as expected. □

Theorem 4.3 (Reduction of Order)**.** *Assume κ_0, κ_1 satisfy* (1.2)*. If y_1 is a solution of the linear homogeneous equation* (4.1)*, in other words if $Ly_1 = 0$, then*

$$y_2(t) = y_1(t) \int_{t_0}^t \frac{1}{p(s)} e_{Y_1}(s, a) d_\alpha s, \qquad Y_1(s) := 2\kappa_1(\alpha, s) - 2\frac{D^\alpha y_1(s)}{y_1(s)}, \tag{4.2}$$

is a second linearly independent solution of (4.1)*, for e_{Y_1} defined as in* (1.5)*.*

Proof. Referring to (4.1), assume y_1 is a solution of $Ly = 0$, and let y_2 take the form (4.2). Using Lemma 1.7, we have

$$\begin{aligned} D^\alpha y_2(t) &= y_1(t) D^\alpha \int_{t_0}^t \frac{1}{p(s)} e_{Y_1}(s, a) d_\alpha s \\ &\quad + (D^\alpha y_1(t) - \kappa_1(\alpha, t) y_1(t)) \int_{t_0}^t \frac{1}{p(s)} e_{Y_1}(s, a) d_\alpha s \end{aligned}$$

$$= \frac{y_1(t)}{p(t)} e_{Y_1}(t,a) + (D^\alpha y_1(t)) \int_{t_0}^{t} \frac{1}{p(s)} e_{Y_1}(s,a) d_\alpha s.$$

Then

$$\begin{aligned} & p(t)\left(D^\alpha y_2(t) - \kappa_1(\alpha,t) y_2(t)\right) \\ = & \; y_1(t) e_{Y_1}(t,a) + p(t)\left(D^\alpha y_1(t) - \kappa_1(\alpha,t) y_1(t)\right) \int_{t_0}^{t} \frac{1}{p(s)} e_{Y_1}(s,a) d_\alpha s, \end{aligned}$$

so that (suppressing the arguments)

$$\begin{aligned} D^\alpha\left[p\left(D^\alpha y_2 - \kappa_1 y_2\right)\right] &= y_1 Y_1 e_{Y_1} + e_{Y_1}\left(D^\alpha y_1 - \kappa_1 y_1\right) \\ & \quad + p\left(D^\alpha y_1 - \kappa_1 y_1\right)\left(\frac{1}{p} e_{Y_1} + \kappa_1 \int_{t_0}^{t} \frac{1}{p(s)} e_{Y_1}(s,a) d_\alpha s\right) \\ & \quad - \left[q y_1 + \kappa_1 p (D^\alpha y_1 - \kappa_1 y_1)\right] \int_{t_0}^{t} \frac{1}{p(s)} e_{Y_1}(s,a) d_\alpha s, \end{aligned}$$

where in the last line we have used the fact that y_1 is a solution. Continuing to simplify, we have

$$\begin{aligned} D^\alpha\left[p\left(D^\alpha y_2 - \kappa_1 y_2\right)\right] &= -q y_2 + \left(y_1\left(2\kappa_1 - 2\frac{D^\alpha y_1}{y_1}\right) + D^\alpha y_1 - \kappa_1 y_1\right) e_{Y_1}, \\ & \quad + p\left(D^\alpha y_1 - \kappa_1 y_1\right)\left(\frac{1}{p} e_{Y_1} + 0\right) \\ &= -q y_2. \end{aligned}$$

where we have used the form of Y_1 in (4.2). □

We now address the question under which circumstances an equation of the form

$$D^\alpha D^\alpha x + a(t) D^\alpha x + b(t) x = 0$$

can be rewritten in self-adjoint form (4.1).

Theorem 4.4. *Assume κ_0, κ_1 satisfy* (1.2). *If κ_1 is differentiable on $[t_0, \infty)$ and $a, b : [t_0, \infty) \to \mathbb{R}$ are continuous functions, then the iterated conformable equation*

$$D^\alpha D^\alpha x + a(t) D^\alpha x + b(t) x = 0, \quad t \in [t_0, \infty), \tag{4.3}$$

can be written in self-adjoint form (4.1)*, where*

$$p(t) = e_{a+2\kappa_1}(t, t_0), \quad q(t) = p(t)\left(\kappa_1(\alpha,t) a(t) + b(t) + D^\alpha \kappa_1(\alpha,t)\right), \tag{4.4}$$

for $t \in [t_0, \infty)$.

Proof. Assume x is a solution of (4.3). By (4.4), after suppressing the arguments, we see that

$$pa = D^\alpha p - 2p\kappa_1, \qquad pb = q - pa\kappa_1 - pD^\alpha \kappa_1 = q - D^\alpha(p\kappa_1) + p\kappa_1^2. \tag{4.5}$$

Multiplying both sides of (4.3) by p and using (4.5), we get

$$\begin{aligned} 0 &= pD^\alpha D^\alpha x + paD^\alpha x + pbx \\ &= pD^\alpha(D^\alpha x) + (D^\alpha p - 2p\kappa_1)D^\alpha x + (q - D^\alpha(p\kappa_1) + p\kappa_1^2)x \\ &= pD^\alpha(D^\alpha x) + (D^\alpha x)(D^\alpha p - p\kappa_1) - p\kappa_1 D^\alpha x + (q - D^\alpha(p\kappa_1) + p\kappa_1^2)x \\ &= D^\alpha\left[pD^\alpha x\right] - D^\alpha\left[p\kappa_1 x\right] + qx. \end{aligned}$$

By the linearity of D^α this equation is in self-adjoint form $Lx = 0$ with p, q given by (4.4). □

Definition 4.5 (Wronskian)**.** Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy (1.2). If $x, y : [t_0,\infty) \to \mathbb{R}$ are differentiable on $[t_0,\infty)$, then the conformable Wronskian of x and y is given by

$$W(x,y)(t) = \det\begin{pmatrix} x(t) & y(t) \\ D^\alpha x(t) & D^\alpha y(t) \end{pmatrix} \quad \text{for} \quad t \in [t_0,\infty), \tag{4.6}$$

for D^α given in (1.3).

Definition 4.6 (Lagrange Bracket)**.** If $x, y : [t_0,\infty) \to \mathbb{R}$ are differentiable on $[t_0,\infty)$, then the Lagrange bracket of x and y is defined by

$$\{x; y\}(t) = p(t)W(x,y)(t) \quad \text{for} \quad t \in [t_0,\infty),$$

where W is the Wronskian given in (4.6).

Theorem 4.7 (Lagrange Identity)**.** *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = D^\alpha\{x; y\}(t) \quad \textit{for} \quad t \in [t_0,\infty).$$

Proof. By the conformable product rule we have

$$\begin{aligned} D^\alpha\{x; y\} &= D^\alpha\left[xp\left(D^\alpha y - \kappa_1 y\right) - yp\left(D^\alpha x - \kappa_1 x\right)\right] \\ &= xD^\alpha\left[p\left(D^\alpha y - \kappa_1 y\right)\right] + p(D^\alpha y - \kappa_1 y)(D^\alpha x - \kappa_1 x) \\ &\quad - yD^\alpha\left[p\left(D^\alpha x - \kappa_1 x\right)\right] - p(D^\alpha x - \kappa_1 x)(D^\alpha y - \kappa_1 y) \\ &= xD^\alpha\left[p\left(D^\alpha y - \kappa_1 y\right)\right] - yD^\alpha\left[p\left(D^\alpha x - \kappa_1 x\right)\right] \\ &= x\left\{qy + D^\alpha\left[p\left(D^\alpha y - \kappa_1 y\right)\right]\right\} - y\left\{qx + D^\alpha\left[p\left(D^\alpha x - \kappa_1 x\right)\right]\right\} \\ &= xLy - yLx \end{aligned}$$

on $[t_0,\infty)$. □

From Theorem 4.7 we have the following two corollaries.

Corollary 4.8 (Abel's Formula)**.** *If x and y both solve* (4.1)*, then*

$$W(x,y)(t) = \frac{ce_0(t,t_0)}{p(t)} \quad \textit{for all} \quad t \in [t_0,\infty),$$

where $c \in \mathbb{R}$ is a constant.

Proof. Assume x and y both solve (4.1). By the Lagrange identity,

$$D^\alpha\{x;y\}(t) = 0 \quad \text{for} \quad t \in [t_0,\infty),$$

whence $\{x;y\}(t)$ must be constant with respect to D^α, that is

$$\{x;y\}(t) = ce_0(t,t_0), \quad t \in [t_0,\infty),$$

for any constant $c \in \mathbb{R}$. □

Corollary 4.9. *If x and y both solve* (4.1)*, then either*

$$W(x,y)(t) \equiv 0 \quad \textit{for all} \quad t \in [t_0,\infty)$$

or

$$W(x,y)(t) \neq 0 \quad \textit{for all} \quad t \in [t_0,\infty),$$

where the first case occurs if and only if x and y are linearly dependent on $[t_0,\infty)$, and the second occurs iff x and y are linearly independent on $[t_0,\infty)$. Note that the results hold for $\alpha \equiv 1$.

Proof. Assume x and y both solve (4.1). By Abel's formula, Corollary 4.8,

$$W(x,y)(t) = \frac{ce_0(t,t_0)}{p(t)}$$

for all $t \in [t_0,\infty)$. If x and y are linearly dependent, clearly $W(x,y)(t) \equiv 0$ for all $t \in [t_0,\infty)$.On the other hand, if $W(x,y)(t) \equiv 0$ for all $t \in [t_0,\infty)$, then

$$0 = xD^\alpha y - yD^\alpha x = x(\kappa_0 y' + \kappa_1 y) - y(\kappa_0 x' + \kappa_1 x) = \kappa_0(xy' - x'y),$$

so that x and y are linearly dependent. □

Remark 4.10. Define the conformable inner product of two continuous functions to be

$$\langle y,z\rangle = \int_a^b y(t)z(t)e_0(b,t)d_\alpha t$$

in terms of the conformable exponential function (1.5) and integral (1.8). Recall the Lagrange bracket of x and y is

$$\{x;y\}(t) = p(t)W(x,y)(t) \quad \text{for} \quad t \in [t_0,\infty),$$

where W is the Wronskian given in (4.6). Integrating the Lagrange identity (Theorem 4.7) and switching to the inner product notation,

$$\begin{aligned}\langle x, Ly\rangle - \langle y, Lx\rangle &= \int_a^b D^\alpha\{x;y\}(t)e_0(b,t)d_\alpha t \\ &= \{x;y\}(b) - \{x;y\}(a)e_0(b,a) \\ &= p(b)W(x,y)(b) - p(a)W(x,y)(a)e_0(b,a).\end{aligned}$$

It follows that equation (4.1) is formally self adjoint with respect to the inner product above, that is, the identity

$$\langle x, Ly\rangle = \langle y, Lx\rangle$$

holds provided that x and y satisfy the self-adjoint boundary conditions at a and b, namely

$$p(b)W(x,y)(b) = p(a)W(x,y)(a)e_0(b,a).$$

The following concept of the Cauchy function will help for the study of self-adjoint equations.

Definition 4.11 (Cauchy Function)**.** A function $x : [t_0,\infty)\times[t_0,\infty) \to \mathbb{R}$ is the Cauchy function for (4.1) provided for each fixed $s \in [t_0,\infty)$, $x(\cdot,s)$ is the solution of the initial value problem

$$Lx(\cdot,s) = 0, \quad x(s,s) = 0, \quad D^\alpha x(s,s) = \frac{1}{p(s)}.$$

It is easy to verify the following example.

Example 4.12. In (4.1), if $q = 0$, then the Cauchy function for

$$D^\alpha\left[p(t)\left(D^\alpha x(t) - \kappa_1(\alpha,t)x(t)\right)\right] = 0$$

is given by

$$x(t,s) = \int_s^t \frac{e_0(\tau,s)}{p(\tau)}d_\alpha\tau$$

for all $t, s \in [t_0,\infty)$. This can be seen by using Lemma 1.10. □

We now state and prove a theorem that gives a formula for the Cauchy function for (4.1).

Theorem 4.13. *If u and v are linearly independent solutions of* (4.1)*, then the Cauchy function $x(t,s)$ for* (4.1) *is given by*

$$x(t,s) = \frac{u(s)v(t) - v(s)u(t)}{p(s)[u(s)D^\alpha v(s) - v(s)D^\alpha u(s)]} \quad \textit{for} \quad t,s \in [t_0,\infty). \tag{4.7}$$

Proof. Let $y(t,s)$ be defined by the right-hand side of equation (4.7). Then note that for each fixed s, $y(\cdot,s)$ is a linear combination of the solutions u and v and as such is a solution of (4.1). Clearly $y(s,s) = 0$. Also note that

$$D^\alpha y(t,s) = \frac{u(s)D^\alpha v(t) - v(s)D^\alpha u(t)}{p(s)[u(s)D^\alpha v(s) - v(s)D^\alpha u(s)]}.$$

Using the definition of the Wronskian (4.6), we get that

$$D^\alpha y(s,s) = \frac{W(u,v)(s)}{p(s)W(u,v)(s)} = \frac{1}{p(s)}.$$

From the uniqueness of solutions of initial value problems (Theorem 4.1) we have that for each fixed s,

$$x(t,s) = y(t,s),$$

which gives us the desired result. □

Theorem 4.14 (Variation of Constants Formula)**.** *Assume f is continuous on $[t_0,\infty)$ and $a \in [t_0,\infty)$. Let $x(t,s)$ be the Cauchy function for* (4.1)*. Then*

$$x(t) = \int_a^t x(t,s)f(s)d_\alpha s, \quad t \in [t_0,\infty)$$

is the solution of the initial value problem

$$Lx = f(t), \quad x(a) = 0, \quad D^\alpha x(a) = 0.$$

Proof. Let $x(t,s)$ be the Cauchy function for (4.1) and set

$$x(t) = \int_a^t x(t,s)f(s)d_\alpha s.$$

Note that $x(a) = 0$. Taking the conformable derivative D^α of x and using Lemma 1.9 (iv), we get that

$$D^\alpha x(t) \;=\; x(t,t)f(t) + \int_a^t D^\alpha[x(t,s)]f(s)d_\alpha s = \int_a^t D^\alpha[x(t,s)]f(s)d_\alpha s,$$

since the Cauchy function satisfies $x(t,t) = 0$. Note that in the integral, D^α denotes the derivative with respect to the first variable t; thus $D^\alpha x(a) = 0$. From

$$p(t)\left(D^\alpha x(t) - \kappa_1(\alpha,t)x(t)\right) = \int_a^t p(t)\left(D^\alpha x(t,s) - \kappa_1(\alpha,t)x(t,s)\right)f(s)d_\alpha s$$

we conclude from Lemma 1.9 (iv) again that

$$\begin{aligned}
&D^{\alpha}\left[p(t)\left(D^{\alpha}x(t)-\kappa_1(\alpha,t)x(t)\right)\right]\\
&=p(t)\left(D^{\alpha}x(t,t)-\kappa_1(\alpha,t)x(t,t)\right)f(t)\\
&\quad+\int_a^t D^{\alpha}\left[p(t)\left(D^{\alpha}x(t,s)-\kappa_1(\alpha,t)x(t,s)\right)\right]f(s)d_{\alpha}s\\
&=f(t)+\int_a^t\left(-q(t)x(t,s)\right)f(s)d_{\alpha}s\\
&=f(t)-q(t)x(t),
\end{aligned}$$

by all of the properties of the Cauchy function. Consequently, $Lx(t)=f(t)$. □

The following corollary follows immediately.

Corollary 4.15. *Assume f is continuous on $[t_0,\infty)$ and $a\in[t_0,\infty)$. Let $x(t,s)$ be the Cauchy function for* (4.1). *Then*

$$x(t)=u(t)+\int_a^t x(t,s)f(s)d_{\alpha}s,\quad t\in[t_0,\infty)$$

is the solution of the initial value problem

$$Lx=f(t),\quad x(a)=A,\quad D^{\alpha}x(a)=B,$$

where A and B are constants, and where u is the solution of the initial value problem $Lu=0$, $u(a)=A$, $D^{\alpha}u(a)=B$.

Theorem 4.16 (Comparison Theorem for IVPs)**.** *Assume the Cauchy function x for* (4.1) *satisfies $x(t,s)\geq 0$ for $t\geq s$. If $u,v\in\mathbb{D}$ are functions satisfying*

$$Lu(t)\geq Lv(t)\ \textit{ for all }\ t\in[a,b],\quad u(a)=v(a),\quad D^{\alpha}u(a)=D^{\alpha}v(a),$$

then

$$u(t)\geq v(t)\quad\textit{for all}\quad t\in[a,b].$$

Proof. If we let u and v be as in the statement of this theorem and set

$$w(t):=u(t)-v(t)\quad\text{for all}\quad t\in[a,b],$$

then

$$h(t):=Lw(t)=Lu(t)-Lv(t)\geq 0\quad\text{for all}\quad t\in[a,b].$$

Consequently w solves the initial value problem

$$Lw(t)=h(t),\quad w(a)=D^{\alpha}w(a)=0.$$

It follows from the variation of constants formula (Theorem 4.14) that

$$w(t)=\int_a^t x(t,s)h(s)d_{\alpha}s\geq 0,$$

completing the proof. □

5 Sturm–Liouville Problems

Let $\alpha \in [0,1]$, and let D^α be as in (1.3). In this section we are concerned with the Sturm–Liouville conformable differential equation

$$D^\alpha \left[p \left(D^\alpha x - \kappa_1(\alpha, \cdot)x\right)\right](t) + \left(\lambda r(t) + q(t)\right) x(t) = 0. \tag{5.1}$$

Throughout we assume that p, q, r are real and continuous functions on $[t_0, \infty)$ with

$$p(t) \neq 0 \quad \text{for all} \quad t \in [t_0, \infty),$$

and $r(t) \geq 0$ is not identically zero on $[t_0, \infty)$. Using (4.1), equation (5.1) can be written as

$$Lx = -\lambda r(t)x.$$

Our main focus here is the Sturm–Liouville problem

$$\begin{aligned} Lx &= -\lambda r(t)x, \\ \zeta x(a) - \beta D^\alpha x(a) &= 0, \\ \gamma x(b) + \delta D^\alpha x(b) &= 0, \end{aligned} \tag{5.2}$$

where $\zeta, \beta, \gamma, \delta$ are real constants satisfying

$$\zeta^2 + \beta^2 > 0, \quad \gamma^2 + \delta^2 > 0.$$

Definition 5.1. The number λ_0 is an eigenvalue for the Sturm–Liouville problem (5.2) if and only if (5.2) with $\lambda = \lambda_0$ has a nontrivial (not identically zero) solution x_0, called the eigenfunction corresponding to λ_0.

Example 5.2. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous and satisfy (1.2), and let $\kappa_1(\alpha, t) \equiv \kappa_1(\alpha)$, a real constant. In (5.2) let $p(t) \equiv 1$, let $\ell > 0$, and set $2\zeta = \kappa_1(\alpha)$. With these choices, find the eigenpairs for the Sturm–Liouville problem (5.2), namely

$$\begin{gathered} D^\alpha D^\alpha x(t) - 2\zeta D^\alpha x(t) + \lambda x(t) = 0, \\ x(0) = 0 = x(\ell). \end{gathered}$$

By Theorem 3.1, solutions take the form $e_m(t, 0)$, where m is a root of the auxiliary equation

$$m^2 - 2\zeta m + \lambda = 0, \qquad m = \zeta \pm \sqrt{\zeta^2 - \lambda}.$$

If $\lambda < \zeta^2$, then

$$x(t) = c_1 e_{\zeta + \sqrt{\zeta^2 - \lambda}}(t, 0) + c_2 e_{\zeta - \sqrt{\zeta^2 - \lambda}}(t, 0),$$

though the boundary conditions at 0 and ℓ require $c_1 = 0 = c_2$, so there are no eigenvalues in this case.

If $\lambda = \zeta^2$, then

$$x(t) = c_1 e_\zeta(t,0) + c_2 e_\zeta(t,0) \int_0^t 1 d_\alpha s,$$

but again the boundary conditions force $c_1 = 0 = c_2$, so there is no eigenvalue in this case.

If $\lambda > \zeta^2$, then

$$x(t) = c_1 e_\zeta(t,0) \cos\left(\int_0^t \sqrt{\lambda - \zeta^2} d_\alpha s\right) + c_2 e_\zeta(t,0) \sin\left(\int_0^t \sqrt{\lambda - \zeta^2} d_\alpha s\right);$$

immediately $x(0) = 0$ implies $c_1 = 0$, and $x(\ell) = 0$ implies

$$\int_0^\ell \sqrt{\lambda - \zeta^2} d_\alpha s = n\pi, \quad n \in \mathbb{N}.$$

Solving for λ yields the eigenvalues

$$\lambda = \lambda_n = \zeta^2 + \left(\frac{n\pi}{\int_0^\ell 1 d_\alpha s}\right)^2, \quad n \in \mathbb{N},$$

with corresponding eigenfunctions

$$x(t) = x_n(t) = e_\zeta(t,0) \sin\left(\frac{n\pi \int_0^t 1 d_\alpha s}{\int_0^\ell 1 d_\alpha s}\right).$$

At $\alpha = 1$ ($\zeta = 0$) the eigenpair reduces to the familiar $(\lambda_n, x_n) = ((n\pi/\ell)^2, \sin(n\pi t/\ell))$.
□

Theorem 5.3. *All eigenvalues of the Sturm–Liouville problem* (5.2) *are real and simple. Corresponding to each eigenvalue there is a real-valued eigenfunction. Eigenfunctions corresponding to distinct eigenvalues of* (5.2) *are orthogonal with respect to the weight function* r *and* e_0 *on* $[a, b]$, *that is to say*

$$\int_a^b x_1(t) x_2(t) r(t) e_0(b,t) d_\alpha t = 0$$

for eigenfunctions x_1 *and* x_2 *corresponding to distinct eigenvalues.*

Proof. The proof is very similar to the classic ($\alpha = 1$) and is thus omitted. □

6 Gronwall Inequality

We begin this section with a comparison theorem, throughout which we let $\alpha \in (0,1]$, $t \in [t_0, \infty)$, and κ_0, κ_1 satisfy (1.2).

Lemma 6.1. *Let p, y, f be continuous functions on $[t_0, \infty)$, and let the exponential function e_p be as in* (1.5). *Then*

$$D^\alpha y(t) \leq p(t)y(t) + f(t) \quad \textit{for all} \quad t \in [t_0, \infty)$$

implies

$$y(t) \leq y(t_0)e_p(t,t_0) + \int_{t_0}^t e_p(t,s)f(s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty).$$

Proof. Using the conformable product rule (Lemma 1.7 (iii)), we see that

$$D^\alpha \left[y(t)e_{\kappa_1 - p}(t,t_0)\right] = [D^\alpha y(t) - p(t)y(t)]e_{\kappa_1 - p}(t,t_0).$$

Multiplication by $e_0(t,s)$ and integration of both sides yields via Lemma 1.9 (ii)

$$\begin{aligned} y(s)e_{\kappa_1-p}(s,t_0)e_0(t,s)\Big|_{s=t_0}^t &= \int_{t_0}^t [D^\alpha y(s) - p(s)y(s)]e_{\kappa_1-p}(s,t_0)e_0(t,s)d_\alpha s \\ y(t)e_{\kappa_1-p}(t,t_0) - y(t_0)e_0(t,t_0) &\leq \int_{t_0}^t f(s)e_{\kappa_1-p}(s,t_0)e_0(t,s)d_\alpha s, \end{aligned}$$

so that

$$y(t) \leq y(t_0)\frac{e_0(t,t_0)}{e_{\kappa_1-p}(t,t_0)} + \int_{t_0}^t f(s)\frac{e_{\kappa_1-p}(s,t_0)e_0(t,s)}{e_{\kappa_1-p}(t,t_0)}d_\alpha s.$$

Now by (1.5) we have that

$$\frac{e_0(t,t_0)}{e_{\kappa_1-p}(t,t_0)} = e_p(t,t_0) \quad \text{and} \quad \frac{e_{\kappa_1-p}(s,t_0)e_0(t,s)}{e_{\kappa_1-p}(t,t_0)} = e_p(t,s).$$

As a result, the assertion follows. □

Theorem 6.2 (Gronwall's Inequality)**.** *Let p, y, f be continuous functions on $[t_0, \infty)$, with $p \geq 0$. Then*

$$y(t) \leq f(t) + \int_{t_0}^t p(s)y(s)e_0(t,s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty)$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t p(s)f(s)e_p(t,s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty).$$

Proof. If we set

$$z(t) = \int_{t_0}^{t} p(s)y(s)e_0(t,s)d_\alpha s,$$

then $z(t_0) = 0$, and by Lemma 1.9 (i) we have

$$D^\alpha z(t) = p(t)y(t) \leq p(t)[f(t) + z(t)] = p(t)f(t) + p(t)z(t).$$

By Lemma 6.1,

$$z(t) \leq \int_{t_0}^{t} e_p(t,s)p(s)f(s)d_\alpha s,$$

and hence the claim follows because of $y(t) \leq f(t) + z(t)$. □

Corollary 6.3. *Let p, y be continuous functions on $[t_0, \infty)$, with $p \geq 0$. Then*

$$y(t) \leq \int_{t_0}^{t} p(s)y(s)e_0(t,s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty)$$

implies

$$y(t) \leq 0 \quad \textit{for all} \quad t \in [t_0, \infty).$$

Proof. This is Theorem 6.2 with $f(t) \equiv 0$. □

Corollary 6.4. *Let p, y be continuous functions on $[t_0, \infty)$ with $p \geq 0$, and let $\delta \in \mathbb{R}$. Then*

$$y(t) \leq \delta + \int_{t_0}^{t} p(s)y(s)e_0(t,s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty)$$

implies

$$y(t) \leq \delta e_p(t,t_0) + \delta \int_{t_0}^{t} \kappa_1(\alpha,s)e_p(t,s)d_\alpha s \quad \textit{for all} \quad t \in [t_0, \infty).$$

Proof. First, note that

$$e_p(t,s) = e_{\kappa_1 - p}(s,t)e_0(t,s). \tag{6.1}$$

If we let $f(t) = \delta$ in Theorem 6.2 and use Theorem 1.7 (v), then by (6.1) we have

$$\begin{aligned}
y(t) &\leq \delta\left[1 + \int_{t_0}^{t} p(s)e_{\kappa_1-p}(s,t)e_0(t,s)d_\alpha s\right] \\
&= \delta\left[1 - \int_{t_0}^{t} \left(\kappa_1(\alpha,s) - p(s)\right) e_{\kappa_1-p}(s,t)e_0(t,s)d_\alpha s\right. \\
&\quad \left. + \int_{t_0}^{t} \kappa_1(\alpha,s)e_{\kappa_1-p}(s,t)e_0(t,s)d_\alpha s\right] \\
&= \delta\left[1 - \int_{t_0}^{t} D^\alpha\left(e_{\kappa_1-p}(s,t)\right)e_0(t,s)d_\alpha s + \int_{t_0}^{t} \kappa_1(\alpha,s)e_p(t,s)d_\alpha s\right]
\end{aligned}$$

$$
\begin{aligned}
&= \delta\left[1 - e_{\kappa_1 - p}(s,t)e_0(t,s)\Big|_{s=t_0}^{s=t} + \int_{t_0}^{t} \kappa_1(\alpha,s)e_p(t,s)d_\alpha s\right] \\
&= \delta\left[e_{\kappa_1 - p}(t_0,t)e_0(t,t_0) + \int_{t_0}^{t} \kappa_1(\alpha,s)e_p(t,s)d_\alpha s\right] \\
&= \delta e_p(t,t_0) + \delta\int_{t_0}^{t} \kappa_1(\alpha,s)e_p(t,s)d_\alpha s.
\end{aligned}
$$

This completes the proof. □

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