Partial Hadamard Fractional Integral Equations

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Abstract

This paper deals with the existence and uniqueness of solutions for a class of partial integral equations via Hadamard’s fractional integral, by applying some fixed point theorems.

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1 Introduction

The fractional calculus represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [11, 17]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [3, 4], Kilbas et al. [12], Miller and Ross [13], and the papers of Abbas et al. [1, 2, 5], Benchohra et al. [6], Vityuk et al. [18, 19], and the references therein.

In [7], Butzer et al. investigated properties of the Hadamard fractional integral and derivative. In [8], they obtained the Mellin transform of the Hadamard fractional integral and differential operators, and in [15], Pooseh et al. obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [16] and the references therein.

This article deals with the existence and uniqueness of solutions to the following Hadamard partial fractional integral equation of the form

\[
\begin{align*}
    u(x, y) &= \mu(x, y) \\
    &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1-1} \left( \log \frac{y}{t} \right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} \, dtds; \text{ if } (x, y) \in J,
\end{align*}
\]

where \( J := [1, a] \times [1, b], a, b > 1, r_1, r_2 > 0, \mu : J \to \mathbb{R}, f : J \times \mathbb{R} \to \mathbb{R} \) are given continuous functions.

We present two results for the integral equation (1.1). The first one is based on Banach’s contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. This paper initiates the study of Hadamard integral equations of two independent variables.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper. We let \( C := C(J, \mathbb{R}) \) be the Banach space of continuous functions \( u : J \to \mathbb{R} \) with the norm

\[
||u||_C = \sup_{(x,y) \in J} |u(x, y)|,
\]

and \( L^1(J, \mathbb{R}) \) be the Banach space of functions \( u : J \to \mathbb{R} \) that are Lebesgue integrable with norm

\[
||u||_{L^1} = \int_1^a \int_1^b |u(x, y)| \, dydx.
\]
Definition 2.1 (See [10, 12]). The Hadamard fractional integral of order \( q > 0 \) for a function \( g \in L^1([1, a], \mathbb{R}) \), is defined as
\[
(HI_q^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left( \log \frac{x}{s} \right)^{q-1} g(s) \frac{ds}{s},
\]
where \( \Gamma(\cdot) \) is the Euler gamma function.

Definition 2.2. Let \( r_1, r_2 \geq 0, \sigma = (1, 1) \) and \( r = (r_1, r_2) \). For \( w \in L^1(J, \mathbb{R}) \), define the Hadamard partial fractional integral of order \( r \) by the expression
\[
(HI_r^\sigma w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1-1} \left( \log \frac{y}{t} \right)^{r_2-1} w(s,t) \frac{dtds}{st},
\]

Theorem 2.3 (See [9]; Nonlinear alternative of Leray–Schauder type). Let \( X \) be a Banach space and \( C \) a nonempty convex subset of \( X \). Let \( U \) a nonempty open subset of \( C \) with \( 0 \in U \) and \( T : U \to C \) be a continuous and compact operator. Then, either
(a) \( T \) has fixed points, or
(b) There exist \( u \in \partial U \) and \( \lambda \in (0, 1) \) with \( u = \lambda T(u) \).

Set \( J_0 := \{(x,y,s) : 0 \leq s \leq x \leq a, y \in [0, b]\}, J_1 := \{(x,y,s,t) : 0 \leq s \leq x \leq a, 0 \leq t \leq y \leq b\}, D_1 := \frac{\partial}{\partial x}, D_2 := \frac{\partial}{\partial y} \) and \( D_1D_2 := \frac{\partial^2}{\partial x\partial y} \).

In the sequel we will make use of the following variant of the inequality for two independent variables due to Pachpatte.

Lemma 2.4 (See [14]). Let \( w \in C(J, \mathbb{R}_+), p, D_1p \in C(J_0, \mathbb{R}_+), q, D_1q, D_2q, D_1D_2q \in C(J_1, \mathbb{R}_+) \) and \( c > 0 \) be a constant. If
\[
w(x,y) \leq c + \int_0^x p(x,y,s)w(s,y)ds + \int_0^x \int_0^y q(x,y,s,t)w(s,t)dtds,
\]
for \( (x,y) \in [0, a] \times [0, b] \), then
\[
w(x,y) \leq cA(x,y) \exp \left( \int_0^x \int_0^y B(s,t)dtds \right),
\]
where
\[
A(x,y) = \exp(Q(x,y)),
\]
in which
\[
Q(x,y) = \int_0^x \left[ p(s,y,s) + \int_0^s D_1p(s,y,\xi)d\xi \right] ds,
\]
and
\[
B(x,y) = q(x,y,x,y)A(x,y) + \int_0^x D_1q(x,y,s,y)A(s,y)ds + \int_0^y D_2q(x,y,x,t)A(x,t)dt + \int_0^x \int_0^y D_1D_2q(x,y,s,t)A(s,t)dtds.
\]
From the above lemma and when \( p \equiv 0 \), we get the following lemma.

**Lemma 2.5.** Let \( w \in C(J, \mathbb{R}_+) \), \( q, D_1q, D_2q, D_1D_2q \in C(J_1, \mathbb{R}_+) \) and \( c > 0 \) be a constant. If
\[
w(x, y) \leq c + \int_1^x \int_1^y q(x, y, s, t)w(s, t)dt\,ds,
\]
for \((x, y) \in J\), then
\[
w(x, y) \leq c \exp \left( \int_1^x \int_1^y B(s, t)dt\,ds \right),
\]
where
\[
B(x, y) = q(x, y, x, y) + \int_1^x D_1q(x, y, s, y)ds
+ \int_1^y D_2q(x, y, x, t)dt + \int_1^x \int_1^y D_1D_2q(x, y, s, t)dt\,ds.
\]

### 3 Main Results

Let us start by defining what we mean by a solution of the integral equation (1.1).

**Definition 3.1.** A function \( u \in C \) is said to be a solution of (1.1) if \( u \) satisfies equation (1.1) on \( J \).

Further, we present conditions for the existence and uniqueness of a solution of the equation (1.1).

**Theorem 3.2.** Assume

\((H_1)\) For any \( u, v \in C \) and \((x, y) \in J\), there exists \( k > 0 \) such that
\[
|f(x, y, u) - f(x, y, v)| \leq k\|u - v\|_C.
\]

If
\[
L := \frac{k(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1, \tag{3.1}
\]
then there exists a unique solution for the equation (1.1) on \( J \).

**Proof.** Transform the integral equation (1.1) into a fixed point equation. Consider the operator \( N : C \to C \) defined by:
\[
(Nu)(x, y) = \mu(x, y)
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1-1} \left( \log \frac{y}{t} \right)^{r_2-1} \frac{f(s, t, u(s, t))}{st}dt\,ds.
\tag{3.2}
\]
Let \( v, w \in C \). Then, for \((x, y) \in J\), we have
\[
| (Nv)(x, y) - (Nw)(x, y) | \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \times \frac{| f(s, t, u(s, t)) - f(s, t, v(s, t)) |}{st} \, dt \, ds
\]
\[
\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \times \frac{k \| u - v \|_C}{st} \, dt \, ds
\]
\[
\leq \frac{k (\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \| v - w \|_C.
\]
Consequently,
\[
\| N(v) - N(w) \|_C \leq L \| v - w \|_C.
\]
By (3.1), \( N \) is a contraction, and hence \( N \) has a unique fixed point by Banach’s contraction principle.

**Theorem 3.3.** Assume that the following hypothesis holds:

\((H_2)\) There exist functions \( p_1, p_2 \in C(J, \mathbb{R}_+) \) such that
\[
| f(x, y, u) | \leq p_1(x, y) + p_2(x, y) | u(x, y) |,
\]
for any \( u \in \mathbb{R} \) and \((x, y) \in J\).

Then the integral equation (1.1) has at least one solution defined on \( J \).

**Proof.** Consider the operator \( N \) defined in (3.2). We shall show that the operator \( N \) is continuous and completely continuous.

**Step 1.** \( N \) is continuous.

Let \( \{ u_n \} \) be a sequence such that \( u_n \to u \) in \( C \). Let \( \eta > 0 \) be such that \( \| u_n \|_C \leq \eta \).

Then
\[
| (Nu_n)(x, y) - (Nu)(x, y) | \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \times \frac{| f(s, t, u_n(s, t)) - f(s, t, u(s, t)) |}{st} \, dt \, ds
\]
\[
\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \times \frac{\sup_{(s,t) \in J} | f(s, t, u_n(s, t)) - f(s, t, u(s, t)) |}{st} \, dt \, ds
\]
\[
\leq \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \times \| f(\cdot, \cdot, u_n(\cdot, \cdot)) - f(\cdot, \cdot, u(\cdot, \cdot)) \|_C.
\]
From Lebesgue’s dominated convergence theorem and the continuity of the function \( f \), we get
\[
|(N u_n)(x, y) - (N u)(x, y)| \to 0 \text{ as } n \to \infty.
\]

**Step 2.** \( N \) maps bounded sets into bounded sets in \( C \).
Indeed, it is enough show that, for any \( \eta^* > 0 \), there exists a positive constant \( \tilde{\ell} \) such that, for each \( u \in B_{\eta^*} = \{ u \in C : \| u \|_C \leq \eta^* \} \), we have \( \| N(u) \|_C \leq \tilde{\ell} \). Set
\[
p^*_i = \sup_{(x,y) \in J} p_i(x, y); \ i = 1, 2.
\]

From \((H_2)\), for each \((x, y) \in J\), we have
\[
|\mu(x, y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \times p_1(s, t) + p_2(s, t)\|u\|_C \, dt \, ds \leq \|\mu\|_C \left( \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} (p^*_1 + p^*_2 \eta^*) \right) \leq \lambda.
\]

Hence
\[
\| N(u) \|_C \leq \tilde{\ell}.
\]

**Step 3:** \( N \) maps bounded sets into equicontinuous sets in \( C \).
Let \((x_1, y_1), (x_2, y_2) \in (1, a) \times (1, b), x_1 < x_2, y_1 < y_2, B_{\eta^*} \) be a bounded set of \( C \) as in Step 2, and let \( u \in B_{\eta^*} \). Then,
\[
|\{(N u)(x_2, y_2) - (N u)(x_1, y_1)\}| \leq |\mu(x_1, y_1) - \mu(x_2, y_2)|
\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[ \log \frac{x_2}{s} \right]^{r_1-1} \left[ \log \frac{y_2}{t} \right]^{r_2-1} \times |f(s, t, u(s, t))| \, dt \, ds
\]
\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[ \log \frac{x_2}{s} \right]^{r_1-1} \left[ \log \frac{y_2}{t} \right]^{r_2-1} \left[ \frac{|f(s, t, u(s, t))|}{st} \right] \, dt \, ds
\]
\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[ \log \frac{x_2}{s} \right]^{r_1-1} \left[ \log \frac{y_2}{t} \right]^{r_2-1} \left[ \frac{|f(s, t, u(s, t))|}{st} \right] \, dt \, ds.
\]
Thus,

\[
|(Nu)(x_2, y_2) - (Nu)(x_1, y_1)| \leq |\mu(x_1, y_1) - \mu(x_2, y_2)|
\]

\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[ \log \frac{x_2}{s} |r_1 - 1| \left| \log \frac{y_2}{t} \right| r_2 - 1 - \log \frac{x_1}{s} |r_1 - 1| \left| \log \frac{y_1}{t} \right| r_2 - 1 \right] \times \frac{p_1^* + p_2^* \eta^*}{st} dtds
\]

\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_2} \int_1^{y_2} \log \frac{x_2}{s} |r_1 - 1| \left| \log \frac{y_2}{t} \right| r_2 - 1 \times \frac{p_1^* + p_2^* \eta^*}{st} dtds
\]

\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_2} \int_1^{y_1} \log \frac{x_2}{s} |r_1 - 1| \left| \log \frac{y_2}{t} \right| r_2 - 1 \times \frac{p_1^* + p_2^* \eta^*}{st} dtds
\]

\[
\leq \frac{p_1^* + p_2^* \eta^*}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \times \left[ 2(\log y_2)^2 (\log x_2 - \log x_1)^{r_1} + 2(\log x_2) \Gamma(1) (\log y_2 - \log y_1)^{r_2} + (\log x_1)^{r_1} (\log y_1)^{r_2} - (\log x_2 - \log x_1) \Gamma(1) (\log y_2 - \log y_1)^{r_2} \right].
\]

As \( x_1 \to x_2 \) and \( y_1 \to y_2 \), the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem, we can conclude that \( N \) is continuous and completely continuous.

**Step 4. (A priori bounds)**

We now show that there exists an open set \( U \subseteq C \) with \( u \neq \lambda N(u) \), for \( \lambda \in (0, 1) \) and \( u \in \partial U \).

Let \( u \in C \) be such that \( u = \lambda N(u) \) for some \( 0 < \lambda < 1 \). Thus, for each \((x, y) \in J\),

\[
u(x, y) = \lambda \mu(x, y) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \log \frac{x}{s} \right)^{r_1 - 1} \left( \log \frac{y}{t} \right)^{r_2 - 1} \frac{f(s, t, u(s, t))}{st} dtds.
\]

This implies that, for each \((x, y) \in J\), we have

\[
|u(x, y)| \leq |\mu(x, y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right| r_1 - 1 \left| \log \frac{y}{t} \right| r_2 - 1 \times \frac{p_1(s, t) + p_2(s, t)|u(s, t)|}{st} dtds
\]

\[
\leq \|\mu\|_\infty + \frac{p_1^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} + \frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right| r_1 - 1 \left| \log \frac{y}{t} \right| r_2 - 1 \frac{|u(s, t)|}{st} dtds.
\]
Thus, for each $(x, y) \in J$, we get

$$|u(x, y)| \leq \|\mu\|_{\infty} + \frac{p_1^*(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} + \frac{p_2^*}{\Gamma(r_1) \Gamma(r_2)} \int_1^x \int_1^y \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1} \frac{|u(s, t)|}{st} dt ds$$

$$\leq c + \int_1^x \int_1^y q(x, y, s, t) |u(s, t)|,$$

where

$$c := \|\mu\|_{\infty} + \frac{p_1^*(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)},$$

and

$$q(x, y, s, t) := \frac{p_2^*}{s t \Gamma(r_1) \Gamma(r_2)} \left| \log \frac{x}{s} \right|^{r_1-1} \left| \log \frac{y}{t} \right|^{r_2-1}.$$

From Lemma 2.5, we obtain

$$|u(x, y)| \leq c \exp \left( \int_1^x \int_1^y B(s, t) dt ds \right),$$

where

$$B(x, y) = q(x, y, x, y) + \int_1^x D_1 q(x, y, s, y) ds$$

$$+ \int_1^y D_2 q(x, y, x, t) dt + \int_1^x \int_1^y D_1 D_2 q(x, y, s, t) dt ds$$

$$\leq \frac{p_2^*}{xy \Gamma(r_1) \Gamma(r_2)} (\log x)^{r_1-1} (\log y)^{r_2-1}.$$ 

Hence

$$|u(x, y)| \leq c \exp \left( \int_1^x \int_1^y \frac{p_2^*}{st \Gamma(r_1) \Gamma(r_2)} (\log s)^{r_1-1} (\log t)^{r_2-1} dt ds \right)$$

$$\leq c \exp \left( \frac{p_2^*(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} \right)$$

$$:= R.$$ 

Set

$$U = \{ u \in C : \|u\|_{\infty} < R + 1 \}.$$

By our choice of $U$, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [9], we deduce that $N$ has a fixed point $u$ in $U$ which is a solution to our equation (1.1).
4 An Example

As an application of our results we consider the following partial Hadamard integral equation of the form

\[ u(x, y) = \mu(x, y) + \int_1^x \int_1^y \left( \frac{\log x}{s} \right)^{r_1-1} \left( \frac{\log y}{t} \right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} \ dt \ ds; \quad (x, y) \in [1, e] \times [1, e], \]  

(4.1)

where

\[ r_1, r_2 > 0, \quad \mu(x, y) = x + y^2; \quad (x, y) \in [1, e] \times [1, e], \]

and

\[ f(x, y, u(x, y)) = \frac{c u(x, y)}{e^{x+y+2}}, \quad (x, y) \in [1, e] \times [1, e], \]

with

\[ c := \frac{e^4}{2} \Gamma(1 + r_1)\Gamma(1 + r_2). \]

For each \( u, \bar{u} \in \mathbb{R} \) and \((x, y) \in [1, e] \times [1, e]\) we have

\[ |f(x, y, u(x, y)) - f(x, y, \bar{u}(x, y))| \leq \frac{c}{e^4} \|u - \bar{u}\|. \]

Hence condition \((H_1)\) is satisfied with \( k = \frac{c}{e^4} \). We shall show that condition (3.1) holds with \( a = b = e \). Indeed,

\[ \frac{k (\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} = \frac{c}{e^4 \Gamma(1 + r_1)\Gamma(1 + r_2)} = \frac{1}{2} < 1. \]

Consequently, Theorem 3.2 implies that the integral equation (4.1) has a unique solution defined on \([1, e] \times [1, e]\).

References


