Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 10, Number 1, pp. 85–96 (2015)
http://campus.mst.edu/adsa

Problems for a Linear Two-Time-Scale Discrete Model

Tahia Zerizer
Jazan University
Department of Mathematics, College of Sciences
Jazan, Saudi Arabia
tzerizer@yahoo.fr

Abstract

In this paper, we study problems for a linear two-time-scale time-variant discrete system named D-model. For the boundary value problem, we give conditions that guarantee existence and uniqueness of the solution and a convergent iterative algorithm to compute asymptotic solutions. We use a *perturbation method*, it consists in writing the solution as a straightforward development in the small parameter without need to compute boundary layer correction terms as for the *singular perturbation method*. We give similar results for the final value problem; for both problems we achieve model order reduction and remove the time scale, and we discuss the initial value problem.

AMS Subject Classifications: 93C05, 93C55, 93C70, 93C73.
Keywords: Singularly perturbed system, discrete-time system, two-time-scale system, boundary value problem.

1 Introduction

The theory of discrete dynamical systems and difference equations developed greatly in the last thirty years, and the application of the theory of singular perturbations and time scales has been a powerful analytical tool in their analysis, see [1, 5, 6, 10]. There are two categories for the singularly perturbed discrete systems, fast sampling rate models and slow sampling rate models. The general form of the slow sampling rate models is given in [5–7] as

$$\begin{pmatrix} x_{k+1} \\ \varepsilon^{2l} y_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11} & \varepsilon^{1-p} A_{12} \\ \varepsilon^{p} A_{21} & \varepsilon A_{22} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad k = 0, \dots, N-1, \tag{1.1}$$

Received April 30, 2014; Accepted September 11, 2014
Communicated by Vera Zeidan

where $l = 0, 1$; $p = 0, 1$; and $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$ are the state vectors at the kth discrete time; A_{ij}, $i, j = 1, 2$, are constant matrices with appropriate dimensions, and ε is a small positive parameter. The three cases of system (1.1) result in the C-model $(l = 0, p = 0)$ and the R-model $(l = 0, p = 1)$, where the small parameter appears respectively, in the column and the row of the system matrix, and the D-model $(l = 1, p = 1)$ where the small parameter is located in an identical fashion to that of the continuous systems described by differential equations. In this paper, we focus on the D-model for the general time-variant case,

$$\begin{pmatrix} x_{k+1} \\ \varepsilon y_{k+1} \end{pmatrix} = \begin{pmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad k = 0, \ldots, N-1, \tag{1.2}$$

i.e., system matrix depends on the discrete time k, and ε is a real parameter. We associate to (1.2), the boundary values

$$x_0 = \alpha, \quad y_N = \beta, \tag{1.3}$$

with α and β are given vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. In optimal control, boundary value problems are frequently encountered, see [8]. The solution of boundary value problems is always a concern. The time-invariant case of problem (1.2)–(1.3) was considered in [7]. The authors used the *singular perturbation method* to give approximate solutions. This formal method uses similar ideas with continuous singularly perturbed system. It consists of finding the approximate solution in terms of an *outer solution* and a *boundary layer correction solution*, see [2, 3, 5–7]. However these authors did not address the question regarding the existence and uniqueness of the solution and gave only the comparison between the full system and the reduced system without proving the accuracy of their approximations. This paper is mainly devoted to the study of the boundary value problem (1.2)–(1.3). We give conditions which guarantee the existence and uniqueness of the solution and we develop the *perturbation method* we proposed for the singularly perturbed difference equations, see [9, 11].

The structure of the paper is as follows. In Section 2, we give the main result. We study the existence and uniqueness of the solution of problem (1.2)–(1.3) and we develop the *perturbation method*. We obtain uniform and straightforward asymptotic approximations, found by a convergent iterative algorithm and we determine the domain of the small parameter thus ensuring the validity of the approximation. Section 3 is devoted to the final value problem associated to the D-model (1.2), we give similar results. We study the existence and uniqueness of the solution and an algorithm to solve the problem. For both problems, we achieve model order reduction and time scale separation. In Section 4, we discuss the initial value problem. The perturbation method is not applicable because the resulting reduced problem may not have a solution. We end our paper with a conclusion in Section 5.

2 Main Result

In this section, we give the main result. The *perturbation method* consists in writing the solution $(x_k(\varepsilon), y_k(\varepsilon))'$, $k = 0, \ldots, N$, as a power series in the small parameter ε:

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \ldots, N. \tag{2.1}$$

In theorem 2.1, we prove the convergence of the series (2.1) and we give conditions that guarantee the existence and uniqueness of the solution of the boundary value problem (1.2)–(1.3). A convergent algorithm is given to indicate the sequence of steps for computing the terms of these series.

2.1 Formal Asymptotic Solution

We seek a straightforward expansion of the form (2.1). Substituting the power series (2.1) into in (1.2)–(1.3) and equating coefficients term-wise then determines the coefficients of (2.1) successively. Thus for the zero*th* order approximation, we must have the following equations

$$x_0^{(0)} = \alpha, \tag{2.2}$$

$$x_{k+1}^{(0)} = A_{11}(k)x_k^{(0)} + A_{12}(k)y_k^{(0)}, \quad k = 0, \ldots, N-1, \tag{2.3}$$

$$0 = A_{21}(k)x_k^{(0)} + A_{22}(k)y_k^{(0)}, k = 0, \ldots, N-1, \tag{2.4}$$

$$y_N^{(0)} = \beta. \tag{2.5}$$

The system (2.2)–(2.3)–(2.4)–(2.5) defines the *reduced problem* of the boundary value problem (1.2)–(1.3), it results from the cancellation of the small parameter ε in the original problem (1.2)–(1.3). The algebraic equation (2.4) sets a relationship between the coefficients $x_k^{(0)}$ and $y_k^{(0)}$. We can write $y_k^{(0)}$ according to $x_k^{(0)}$ provided the matrices $A_{22}(k)$, $k = 0, \ldots, N-1$, are nonsingular. Therefore, we have

$$y_k^{(0)} = -A_{22}(k)^{-1}A_{21}(k)x_k^{(0)}, \quad k = 0, \ldots, N-1. \tag{2.6}$$

Substituting (2.6) into (2.3), we find

$$x_{k+1}^{(0)} = \left(A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)\right) x_k^{(0)}, \quad k = 0, \ldots, N-1. \tag{2.7}$$

The problem (2.2)–(2.7) is an initial value problem whose solution is as follows

$$x_0^{(0)} = \alpha, \quad x_k^{(0)} = \prod_{i=0}^{k-1} \left(A_{11}(i) - A_{12}(i)A_{22}(i)^{-1}A_{21}(i)\right) \alpha, \quad k = 1, \ldots, N. \tag{2.8}$$

Thus, from (2.6) and (2.8), we have

$$y_0^{(0)} = -A_{22}(0)^{-1}A_{21}(0)\alpha, \tag{2.9}$$

$$y_k^{(0)} = -A_{22}(k)^{-1}A_{21}(k)\prod_{i=0}^{k-1}\left(A_{11}(i) - A_{12}(i)A_{22}(i)^{-1}A_{21}(i)\right)\alpha, \tag{2.10}$$
$$k = 1, \ldots, N-1.$$

Notice that the terms $y_k^{(0)}$, $k = 0, \ldots, N-1$, can be computed without any knowledge of the final condition $y_N = \beta$. By analogy with the differential equations, we say that there is boundary layer at y_N.
For higher order approximation j, $j \geq 1$, we have the following equations

$$x_0^{(j)} = 0, \tag{2.11}$$

$$x_{k+1}^{(j)} = A_{11}(k)x_k^{(j)} + A_{12}(k)y_k^{(j)}, \quad k = 0, \ldots, N-1, \tag{2.12}$$

$$y_{k+1}^{(j-1)} = A_{21}(k)x_k^{(j)} + A_{22}(k)y_k^{(j)}, \quad k = 0, \ldots, N-1, \tag{2.13}$$

$$y_N^{(j)} = 0. \tag{2.14}$$

In the algebraic equation (2.13), to compute $x_k^{(j)}$, $y_k^{(j)}$, $k = 0, \ldots, N-1$, we need to have at our disposal the values $y_k^{(j-1)}$, $k = 1, \ldots, N$, which are computed from the previous development of order $j-1$. If the matrices $A_{22}(k)$, $k = 0, \ldots, N-1$, are nonsingular, we can write

$$y_k^{(j)} = A_{22}(k)^{-1}y_{k+1}^{(j-1)} - A_{22}(k)^{-1}A_{21}(k)x_k^{(j)}, \quad k = 0, \ldots, N-1. \tag{2.15}$$

According to (2.12) and (2.15), we have for $k = 0, \ldots, N-1$,

$$x_{k+1}^{(j)} = \left(A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)\right)x_k^{(j)} + A_{12}(k)A_{22}(k)^{-1}y_{k+1}^{(j-1)}. \tag{2.16}$$

Notice that the time scale is removed for all subsystems defined above and their dimensions are equal to n or m, lower than the dimension of the original problem equal to $n+m$.

2.2 Convergence of the Asymptotic Solution

We prove that, there is some domain of the small parameter ε, where the power series (2.1) are convergent. Therefore, the approximations defined in section 2.1 are validated.
Suppose

$$v = (x_0, y_0, x_1, y_1, \ldots, x_N, y_N)', \tag{2.17}$$

where the prime denotes the transpose.
We consider for the vector v the norm in $\mathbb{R}^{(n+m)(N+1)}$

$$\|v\| = \max\left(|x_0|, |y_0|, |x_1|, |y_1|, \ldots, |x_N|, |y_N|\right),$$

and for a matrix $A = (a_{ij})$, the associated matrix norm

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \max_{k=0,\ldots,(n+m)(N+1)} \left(\sum_{j=0}^{(n+m)(N+1)} |a_{ij}| \right).$$

Theorem 2.1. *Assume that $A_{22}(k)$, $k = 0, \ldots, N-1$, are nonsingular. There exists a positive real number ε_0, for all ε such that $|\varepsilon| < \varepsilon_0$, the solution of the boundary value problem* (1.2)–(1.3)*, that is $(x_k(\varepsilon), y_k(\varepsilon))'$, $0 \le k \le N$, exists and is unique, and satisfies* (2.1) *uniformly for $0 \le k \le N$, where the terms $x_k^{(0)}$, $y_k^{(0)}$, and $x_k^{(j)}$, $y_k^{(j)}$, $j \ge 1$, for $0 \le k \le N$, are respectively the solutions of* (2.8), (2.9)–(2.10)–(2.5) *and* (2.11)–(2.16), (2.15)–(2.14).
More precisely, for all $k = 0, \ldots, N$, we have

$$\begin{aligned} |x_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j x_k^{(j)}| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \\ |y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)}| &\le C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \end{aligned} \tag{2.18}$$

where C is a positive constant.

Proof. We write the system (2.2)–(2.3)–(2.4)–(2.5) in the matrix form

$$A_0 v^{(0)} = f, \tag{2.19}$$

where $v^{(0)}$ and f are vectors in $\mathbb{R}^{(n+m)(N+1)}$ defined by

$$v^{(0)} := \left(x_0^{(0)}, y_0^{(0)}, x_1^{(0)}, y_1^{(0)}, \ldots, x_N^{(0)}, y_N^{(0)} \right)', \tag{2.20}$$

$$f := (\alpha, 0, 0, \ldots, 0, \beta)', \tag{2.21}$$

and A_0 is the block matrix given below

$$\begin{pmatrix} I_n & 0 & 0 & 0 & \ldots & & & & 0 \\ A_{11}(0) & A_{12}(0) & -I_n & 0 & & & & & \\ A_{21}(0) & A_{22}(0) & 0 & 0 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & \ldots & A_{11}(N-1) & A_{12}(N-1) & -I_n & 0 \\ & & & & \ldots & A_{21}(N-1) & A_{22}(N-1) & 0 & 0 \\ 0 & & & & \ldots & 0 & 0 & 0 & I_m \end{pmatrix}.$$

Employing the Leibniz formula for determinant of block matrices, see [4], we have

$$\det A_0 = \prod_{k=0}^{N-1} \det A_{22}(k).$$

Therefore, if the matrices $A_{22}(k)$, $k = 0, \dots, N-1$, are nonsingular, the matrix A_0 is also nonsingular. Hence, we can denote

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|}, \qquad C := \|A_0^{-1}\| \|f\|. \tag{2.22}$$

The system (2.11)–(2.12)–(2.13)–(2.14) has the form

$$A_0 v^{(j)} = -Uv^{(j-1)}; \quad v^{(j)} := \left(x_0^{(j)}, y_0^{(j)}, x_1^{(j)}, y_1^{(j)}, \dots, x_N^{(j)}, y_N^{(j)}\right)', \tag{2.23}$$

where U is the matrix given below

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & & & & 0 \\ 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & -I_m & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & 0 & 0 & 0 & -I_m \\ 0 & & & & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that the problem (1.2)–(1.3) can be represented in the matrix form

$$A_\varepsilon v = f, \tag{2.24}$$

where v and f are given by (2.17) and (2.21) respectively, and A_ε satisfies

$$A_\varepsilon = A_0 + \varepsilon U.$$

As $|\varepsilon| < \varepsilon_0$, from (2.22) we have $\|\varepsilon U A_0^{-1}\| < 1$. Thus we can write

$$A_0^{-1} \sum_{j=0}^{\infty} \left(-\varepsilon U A_0^{-1}\right)^j = A_0^{-1} \left(I + \varepsilon U A_0^{-1}\right)^{-1} = A_\varepsilon^{-1}. \tag{2.25}$$

Hence, the solution of system (2.24) exists, is unique and satisfies

$$v(\varepsilon) = A_\varepsilon^{-1} f. \tag{2.26}$$

From (2.23), (2.25) and (2.26), it is deduced that

$$v(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^{(j)} v^{(j)}; \quad v^{(j)} = A_0^{-1} \left(-UA_0^{-1}\right)^j f. \tag{2.27}$$

From (2.19), (2.20), (2.21) and (2.23), we verify that the components $x_k^{(0)}$,$y_k^{(0)}$, and $x_k^{(j)}$, $y_k^{(j)}$ are the solutions of the problems (2.2)–(2.3), (2.4)–(2.5) and (2.11)–(2.12), (2.13)–(2.14), respectively. Notice that equations (2.3), (2.4) and (2.12), (2.13) are respectively equivalent to (2.8), (2.9)–(2.10), (2.11)–(2.16), and (2.15)–(2.14), thereby completing the first part of the proof.

To evaluate the reminder of the series, we have

$$\begin{aligned} &\|A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^{\infty} \left(-\varepsilon U A_0^{-1}\right)^{(j)} \| \leq A_0^{-1}\| \sum_{j=n+1}^{\infty} \|\varepsilon U A_0^{-1}\|^j \\ &= \frac{\|A_0^{-1}\| \|\varepsilon U A_0^{-1}\|^n}{1 - \|\varepsilon U A_0^{-1}\|} \leq \|A_0^{-1}\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \tag{2.28}$$

From (2.25) and (2.28), follows

$$\begin{aligned} &\|v(\varepsilon) - \sum_{j=0}^{\infty} \varepsilon^j v^{(j)}\| \leq \|A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^{n} \left(-\varepsilon U A_0^{-1}\right)^j \| \|f\| \\ &\leq \|A_0^{-1}\| \|f\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \tag{2.29}$$

The chosen norm and (2.22) give (2.18). This completes the proof. □

2.3 Algorithm

In the following, we give a recursive convergent algorithm to exhibit the sequence of steps that gives the approximate solutions of problem (1.2)–(1.3).

***Zeroth*-order solution**

- Step 1. Fix $x_0^{(0)} = \alpha$; for $k = 1, \ldots, N$, compute $x_k^{(0)}$, from (2.8). Fix $y_N^{(0)} = \beta$; compute $y_0^{(0)}$ from (2.9), and $y_k^{(0)}$ for $k = 1, \ldots, N-1$, from (2.10).

***First*-order solution**

- Step 2. Fix $x_0^{(1)} = 0$; compute $x_k^{(1)}$ for $k = 1, \ldots, N$, from (2.16), where $y_k^{(0)}$, $k = 1, \ldots, N$, are determined in step 1.
- Step 3. Fix $y_N^{(1)} = 0$; compute $y_k^{(1)}$ for $k = 0, \ldots, N-1$, from (2.15), where $x_k^{(1)}$ are determined for $k = 0, \ldots, N-1$, from step 2.

***J*th-order solution**

- Step 4. Fix $x_0^{(j)} = 0$; compute $x_k^{(j)}$ for $k = 1, \ldots, N$, from (2.16), where $y_k^{(j-1)}$, $k = 1, \ldots, N$, are determined in a previous step.
- Step 5. Fix $y_N^{(j)} = 0$; compute $y_k^{(j)}$ for $k = 0, \ldots, N-1$, from (2.15), where $x_k^{(j)}$, $k = 0, \ldots, N-1$, are determined in a previous step.

  Then determine $\left(x_k^{(0)}, y_k^{(0)}\right)' + \varepsilon \left(x_k^{(1)}, y_k^{(1)}\right)' + \ldots + \varepsilon^j \left(x_k^{(j)}, y_k^{(j)}\right)'$, $k = 0, \ldots, N$.

3 Final Value Problem

In this section, we study the final value problem. We associate to the system (1.2) the values

$$x_N = \alpha, \quad y_N = \beta. \tag{3.1}$$

The time-invariant case of problem (1.2)–(3.1) was considered in [7] . The authors used the heuristic *singular perturbation method* which consists in writing the solution as a sum of an *outer solution* and *boundary layer correction* term. In order to develop the *perturbation method* for the final value problem (1.2)–(3.1), we write the solution as a straightforward development, a power series of the small parameter

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \ldots, N. \tag{3.2}$$

In this section, we give an algorithm for computing the coefficients of the series (3.2), and we give conditions assuring the existence of the solution.

3.1 Perturbation Method

Substituting the power series (3.2) into the system (1.2)–(3.1), we must have for the zero*th* order approximation

$$x_N^{(0)} = \alpha, \quad y_N^{(0)} = \beta, \tag{3.3}$$

and the same equations (2.3) and (2.4) given in Section 2.1 for the boundary value problem (1.2)–(1.3). The system (2.3)–(2.4)–(3.3) is the *reduced problem* of the final value problem (1.2)–(3.1).
It is necessary that the matrices $A_{22}(k)$, $k = 0, \ldots, N-1$, are nonsingular to deduce from (2.3) and (2.4) the equation (2.6) and the following equation (2.7)

$$x_{k+1}^{(0)} = \left(A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)\right) x_k^{(0)}, \quad k = 0, \ldots, N-1.$$

Moreover, if the matrices $A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)$, $k = 0, \ldots, N-1$, are nonsingular, we have from (2.7)

$$x_k^{(0)} = \left(A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)\right)^{-1} x_{k+1}^{(0)}, \quad k = 0, \ldots, N-1. \tag{3.4}$$

For an abbreviated writing, we denote

$$B(k) = A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k), \quad k = 0, \ldots, N-1. \tag{3.5}$$

Solving the final value problem (3.3)–(3.4), follows the solution

$$x_{N-k}^{(0)} = \prod_{i=N-k}^{N-1} B(i)^{-1}\alpha \quad k = 1, \ldots, N. \tag{3.6}$$

Consequently, from (2.6) and (3.6), we have for $k = 1, \dots, N$,

$$y_{N-k}^{(0)} = -A_{22}(N-k)^{-1}A_{21}(N-k)\prod_{i=N-k}^{N-1} B(i)^{-1}A_{21}(i)^{-1}\alpha. \tag{3.7}$$

For higher order approximation j, $j \geq 1$, we must have

$$x_N^{(j)} = 0, \quad y_N^{(j)} = 0, \tag{3.8}$$

and the equations (2.12), (2.13) given in Section 2.1. If for $k = 0, \dots, N-1$, $A_{22}(k)$ are nonsingular, the coefficients of (3.2) satisfy the equations (2.15), and (2.16) given in Section 2.1. In addition, if $B(k)$, $k = 0, \dots, N-1$, defined in (3.5) are nonsingular, we have

$$x_k^{(j)} = B(k)^{-1}x_{k+1}^{(j)} - B(k)^{-1}A_{12}(k)A_{22}(k)^{-1}y_{k+1}^{(j-1)}, \quad k = 0, \dots, N-1. \tag{3.9}$$

We remind the equation (2.15), it is

$$y_k^{(j)} = A_{22}(k)^{-1}y_{k+1}^{(j-1)} - A_{22}(k)^{-1}A_{21}(k)x_k^{(j)}, \quad k = 0, \dots, N-1. \tag{3.10}$$

It is easy to prove the following theorem.

Theorem 3.1. *Assume that $A_{22}(k)$ and $B(k)$, $k = 0, \dots, N-1$, are nonsingular. There exists a positive real number ε_0, for all ε such that $|\varepsilon| < \varepsilon_0$, the solution of the final value problem* (1.2)–(3.1)*, that is $(x_k(\varepsilon), y_k(\varepsilon))'$, $0 \leq k \leq N$, satisfies* (3.2) *uniformly for $0 \leq k \leq N$, where $x_k^{(0)}$, $y_k^{(0)}$, and $x_k^{(j)}$, $y_k^{(j)}$ are the solutions of* (3.3), (3.6), (3.7), *and* (3.8), (3.9), (3.10), *respectively.*
More precisely, for all $0 \leq k \leq N$, we have

$$|x_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j x_k^{(j)}| \leq C\frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0},$$
$$|y_k(\varepsilon) - \sum_{j=0}^{n} \varepsilon^j y_k^{(j)}| \leq C\frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0},$$

where C is a positive constant.

Proof. The proof is similar to that of theorem 2.1.
Briefly, we write the problem (1.2)–(3.1) in the matrix form

$$A_\varepsilon v = f,$$

where

$$v = (x_0, y_0, x_1, y_1, \dots, x_N, y_N)',$$
$$f = (0, 0, \dots, 0, \alpha, \beta)'.$$

The matrix A_ε is the combination $A_0 + \varepsilon U$, where A_0 is the following block matrix

$$\begin{pmatrix} A_{11}(0) & A_{12}(0) & -I_n & 0 & \dots & & & & 0 \\ A_{21}(0) & A_{22}(0) & 0 & 0 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & \dots & A_{11}(N-1) & A_{12}(N-1) & -I_n & 0 \\ & & & & \dots & A_{21}(N-1) & A_{22}(N-1) & 0 & -I_m \\ & & & & \dots & & & I_n & 0 \\ 0 & & & & \dots & 0 & 0 & 0 & I_m \end{pmatrix},$$

and U the matrix below

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & & & & 0 \\ 0 & 0 & 0 & -I_m & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & 0 & 0 & 0 & -I_m \\ & & & & & 0 & 0 & 0 & 0 \\ 0 & & & & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Leibniz formula for the determinant of block matrix, see [4], gives

$$\det A_0 = \prod_{k=0}^{N-1} \det A_{22}(k) \det(A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)).$$

Because we assumed $A_{22}(k)$ and $A_{11}(k) - A_{12}(k)A_{22}(k)^{-1}A_{21}(k)$, $0 \le k \le N-1$, to be nonsingular, hence the matrix A_0 given above is nonsingular. We denote

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|}, \qquad C := \|A_0^{-1}\|\|f\|.$$

The following of the proof is routine and left to the reader. □

3.2 Algorithm

***Zeroth*-order approximation**

- Step 1. Fix $x_N^{(0)} = \alpha$, $y_N^{(0)} = \beta$; compute $x_{N-k}^{(0)}$, $1 \le k \le N$, from (3.6), and $y_{N-k}^{(0)}$, $1 \le k \le N$ from (3.7).

***J*th-order approximation**

- Step 2. Fix $x_N^{(j)} = 0$; compute $x_k^{(j)}$, $0 \le k \le N-1$, from (3.9), where $x_k^{(j-1)}$ and $y_k^{(j-1)}$ are determined from the development of order $j-1$, i.e., a previous step.

- Step 3. Fix $y_N^{(j)} = 0$; compute $y_k^{(j)}$, $0 \leq k \leq N-1$, from (3.10) where the terms $x_k^{(j)}$ are determined in step 2, and $y_{k+1}^{(j-1)}$ from the development of order $j-1$, i.e., a previous step.
  Then determine
$$\left(x_k^{(0)}, y_k^{(0)}\right)' + \varepsilon \left(x_k^{(1)}, y_k^{(1)}\right)' + \ldots + \varepsilon^j \left(x_k^{(j)}, y_k^{(j)}\right)'.$$

4 Initial Value Problem

For the initial value problem, we associate to the system (1.2) the fixed values

$$x_0 = \alpha, \quad y_0 = \beta. \tag{4.1}$$

The resulting reduced problem is the following

$$x_0^{(0)} = \alpha, \tag{4.2}$$

$$x_{k+1}^{(0)} = A_{11}(k)x_k^{(0)} + A_{12}(k)y_k^{(0)}, \quad k = 0, \ldots, N-1, \tag{4.3}$$

$$0 = A_{21}(k)x_k^{(0)} + A_{22}(k)y_k^{(0)}, k = 0, \ldots, N-1, \tag{4.4}$$

$$y_0^{(0)} = \beta. \tag{4.5}$$

In equation (4.4), for $k = 0$, we verify

$$0 = A_{21}(0)x_0^{(0)} + A_{22}(0)y_0^{(0)}, \tag{4.6}$$

thus from (4.2), (4.5) and (4.6) results the equation

$$0 = A_{21}(0)\alpha + A_{22}(0)\beta, \tag{4.7}$$

this condition is not necessary verified. There is an other difficulty, the equation (4.4) is not defined for $k = N$, i.e., there is no indication for the computation of the value $y_N^{(0)}$, it can be optional. The *perturbation method* is not applicable for the initial value problem.

5 Conclusion

In this paper, we have consider a class of nonautonomous discrete singularly perturbed systems said D-model. We studied three problems. For both boundary value problem and final value problem, we have develop the *perturbation method*. Besides the advantage of removing the time scale and the decomposition of the full system in subsystems of reduced order with decoupled state variables, there is no need to compute the *boundary layer correction* terms as for the *singular perturbation method*. Convergent iterative algorithms have been provided showing the steps of the method. For the initial value problem, the resulting reduced problem does not necessary satisfy the given initial values, thus the *perturbation method* is not applicable. Notice, the *perturbation method* can be extended to all important classes of linear singularly perturbed problems resulting from optimal control. This will be indicated separately.

References

[1] H. Kando and T. Iwazumi. Initial Value Problems of Singularly Perturbed Discrete Systems via Time-Scale decomposition, *Int. J. Systems Sci.*, Vol 14 pp. 555–570, 1983.

[2] G. Kishore Babu and M. S. Krishnarayalu. Some Applications of Discrete One Parameter Singular Perturbation Method, *Journal of Control Engineering and Technology*, Vol 4, pp 76–81, 2014.

[3] M. S. Krishnarayalu. Singular Perturbation Methods for a Class of Initial and Boundary Value problems in Multi-parameter Classical Digital Control Systems, *The ANZIAM Journal*, Vol. 46, 67–77, 2004.

[4] Carl D. Meyer. Matrix Analysis and Applied Linear Algebra, Siam books, 2000.

[5] D. S. Naidu. Singular Perturbations and Time Scales in Control Theory and Applications: an Overview *Discrete and Impulsive Systems*, Series B: Applications and Algorithms 9, 233–278, 2002.

[6] D. S. Naidu and D. B. Price. Singular Perturbation and Time Scale Approaches in Discrete Control Systems, *Journal of Guidance*, *Control and Dynamics*, Vol. 11, No. 6 , pp. 592–594, 1988.

[7] D. S. Naidu and A. K. Rao. Singular Perturbation Analysis of Discrete Control Systems, *Lecture Notes in Mathematics*, Springer, Berlin, 1154, 1985.

[8] A. P. Sage and G. C. White III. Optimum System Control, *Prentice Hall, Englewood Cliffs*, 1997.

[9] T. Sari and T. Zerizer. Perturbations for Linear Difference Equations, *Journal of Mathematical Analysis and Applications*, 1, 4352, 2005.

[10] W.T. van Horssen and M.C. ter Brake. On the multiple scales perturbation method for difference equations, *Nonlinear Dyn*, 55: 401–418, 2009.

[11] T. Zerizer. Perturbation Method for Linear Difference Equations with Small Parameters, *Advances in Difference Equations* Article ID 19214, Pages 1–12, 2006.