Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 10, Number 1, pp. 57–75 (2015)
http://campus.mst.edu/adsa

Approximate Controllability of Semilinear Nonautonomous Systems in Hilbert Spaces

Hugo Leiva
Universidad de Los Andes, Departamento de Matematica
Merida, 5101, Venezuela, hleiva@ula.ve

Nelson Merentes
Universidad Central de Venezuela, Departamento de Matematica
Caracas, 1051, Venezuela, nmerucv@gmail.com

Jose Luis Sanchez
Universidad Central de Venezuela, Departamento de Matematica
Caracas, 1051, Venezuela, casanay085@hotmail.com

Ambrosio Tineo Moya
Universidad de Los Andes, Departamento de Matematica
Merida, 5101, Venezuela, atemoya@ula.ve

Abstract

In this paper we give a necessary and sufficient conditions for approximate controllability of a wide class of semilinear nonautonomous systems in Hilbert spaces. This is done by employing skew-product semi-flows technique. As an application we prove the approximate controllability of a broad class of nonautonomous semilinear reaction diffusion equations which includes the semilinear heat equation.

AMS Subject Classifications: 93B05, 93C10.
Keywords: Approximate controllability of semilinear nonautonomous systems, skew-product semi-flows, evolution operators.

1 Introduction

In this paper we study the approximate controllability of the following nonautonomous evolutionary system

$$z' = A(\theta \cdot t)z + B(\theta \cdot t)u + F(\theta \cdot t, z, u(t)), \;\; t \in [0,\tau], \quad \theta \in \Theta, \quad z(t) \in Z, \quad (1.1)$$

Received September 30, 2013; Accepted January 24, 2014
Communicated by Adina Luminiţa Sasu

where the control u belongs to $L^2(0,\tau;U)$, and Z, U are Hilbert spaces. Here Θ is a compact topological Hausdorff space which is invariant under the flow $\sigma(\theta,t) = \theta\cdot t$, for all $\theta \in \Theta$, $t \in [0,\tau]$, $A(\theta)$ is the generator of a strongly continuous compact evolution operator $\Phi(\theta,t)$ in Z, with common domain $D(A(\theta)) = D$, $B(\theta) \in L(U,Z)$ and the mapping $\theta \to B(\theta)z$ is continuous in θ for all z fixed. $L(U,Z)$ is the Banach space of bounded linear operators from U to Z, in particular $L(Z) = L(Z,Z)$. On the other hand, we assume that the linear equation

$$z' = A(\theta\cdot t)z, \quad z \in Z, \quad \theta \in \Theta, \quad t \geq 0, \tag{1.2}$$

generates a linear skew-product semiflow $\pi = (\Phi,\sigma)$ on $\mathcal{E} = Z \times \Theta$ according to Definition 2.1, given by

$$\pi(z,\theta,t) = (\Phi(z,\theta,t), \theta\cdot t), \quad t \geq 0, \tag{1.3}$$

where $\Phi(\theta,t)$ is the evolution operator associated with (1.2), such that $\Phi(\theta,0) = I$, the identity operator in Z. This work has been motivated by the works [2, 30, 31] where the authors study nonautonomous evolution equations using skew-product semi-flow technique.

The following hypothesis will be assumed: The nonlinear function $F : \Theta\times Z\times U \to Z$ is smooth enough and there are $a, b \in \mathbb{R}$ and $\dfrac{1}{2} \leq \beta < 1$ such that

$$\|F(\theta,z,u)\|_Z \leq a\|u\|^\beta + b, \ \ \forall u \in U, z \in Z, \theta \in \Theta, \tag{1.4}$$

and

$$z' = A(\theta\cdot t)z + B(\theta\cdot t)u(t), \tag{1.5}$$

is approximate controllable.

Remark 1.1. The condition (1.4) can be replaced by the following more general one: There are $a, b, c \in \mathbb{R}$ and $\dfrac{1}{2} \leq \beta < 1$ such that

$$\|F(\theta,z,u) - cz\|_Z \leq a\|u\|^\beta + b, \ \ \forall u \in U, z \in Z, \theta \in \Theta,$$

which implies

$$\|F(\theta,z,u)\|_Z \leq a\|u\|^\beta + c\|z\| + b, \ \ \forall u, z \in Z, \theta \in \Theta.$$

The approximate controllability of semi-linear autonomous evolution equations has been studied by several authors, but the nonlinear term $F(z)$ depends only on the spatial variable z and is bounded or sub-linear at infinity. To mention some of them, we have the work done in, e.g., [10, 11, 24–26, 28, 29]. The approximate controllability of the heat equation under a nonlinear perturbation $f(z)$ independent of the variables t and u

$$\begin{cases} z_t = \Delta z + 1_\omega u(t,x) + f(z) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

has been studied by several authors, particularly in [12–14], depending on conditions impose to the nonlinear term $f(z)$. For instance, in [13, 14] the approximate controllability of the system (1.6) is proved if $f(z)$ is sublinear at infinity, i.e.,

$$|f(z)| \leq d|z| + e. \tag{1.7}$$

Also, in the above reference, they mentioned that when f is superlinear at the infinity, the approximate controllability of system the (1.6) fails. However, to our knowledge, the approximate controllability of nonautonomous evolution equations in Hilbert spaces has been little studied,we can mention the papers [2, 30, 31].

Now, we shall describe the strategy of this work: First, we observe that the approximate controllability of the linear system (1.5) is equivalents to the controllability operator has dense range, which allows us to find an approximate right inverse of such operator. After that, we observe that the approximate controllability of the semilinear system (1.1) is equivalent to the semilinear controllability operator to has dense range. Finally, the approximate controllability of the system (1.1) follows from the approximate controllability of (1.5), the compactness of the evolution operator $\Phi(\theta, t)$ generated by the operator $A(\theta)$, the bound (1.4) satisfied by the nonlinear term F and applying Rothe's fixed point Theorem.

In order to reach our goal, we shall use the following preliminaries results:

Proposition 1.2 (See [4, 7]). *Let $\pi = (\Phi, \sigma)$ be linear skew-product semiflows on $\mathcal{E}$. Then there exist constants $M \geq 1$, $W > 0$ such that*

$$\|\Phi(\theta, t)\| \leq Me^{Wt}, \quad \theta \in \Theta, \quad t \in \mathbb{R}^+.$$

Proposition 1.3. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$ and $1 \leq q < r < \infty$. Then $L_r(\mu) \subset L_q(\mu)$ and*

$$\|f\|_q \leq \mu(X)^{\frac{r-q}{rq}} \|f\|_r, \ \ f \in L_r(\mu). \tag{1.8}$$

Proof. By putting $p = \dfrac{r}{q} > 1$ and considering the relation

$$\int_X (|f|^q)^p d\mu = \int_X |f|^r d\mu, \ \ \forall f \in L_r(\mu)$$

the proof follows from [3, Theorem I.V.6]. □

Theorem 1.4 (See [2]). *If $\Phi(\theta, t)x_0$ is weakly continuous in t, uniformly in θ, then for each $\theta \in \Theta$, the function $\Phi(\theta, \cdot)$ is strongly continuous in $[0, +\infty)$.*

Corollary 1.5 (See [2]). *Let X be a reflexive Banach space. Then for each $\theta \in \Theta$ fixed, the function $\Phi^\star(\theta, \cdot)$ is strongly continuous in $[0, +\infty)$.*

Lemma 1.6 (See [8, Lemma 3.14, p. 62])**.** *Let* $\{\alpha_j\}_{j\geq 1}$ *and* $\{\beta_{i,j} : i = 1, 2, \ldots, m\}_{j\geq 1}$ *be two sequences of real numbers such that:* $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. *Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j} = 0, \ \ \forall t \in [0, \tau], \ \ i = 1, 2, \cdots, m$$

iff

$$\beta_{i,j} = 0, \ \ i = 1, 2, \cdots, m; j = 1, 2, \cdots, \infty.$$

Theorem 1.7 (Rothe's Theorem, [16, page 129])**.** *Let* $E(\tau)$ *be a Hausdorff topological vector space. Let* $B \subset E$ *be a closed convex subset such that the zero of* E *is contained in the interior of* B. *Let* $\Phi : B \to E$ *be a continuous mapping with* $\Phi(B)$ *relatively compact in* E *and* $\Phi(\partial B) \subset B$ *(*∂B *denotes the boundary of the set* B*). Then there is a point* $x^* \in B$ *such that* $\Phi(x^*) = x^*$.

2 Controllability of Linear Systems

In this section we characterize the approximate controllability of the linear system (1.5). To this end, notice that for all arbitrary $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = A(\theta \cdot t)z + B(\theta \cdot t)u(t), \, \theta \in \Theta, \ z \in Z, \, t \in [0, \tau], \\ z(0) = z_0, \end{cases} \tag{2.1}$$

admits only one mild solution given by the formula

$$z(t) = \Phi(\theta, t)z_0 + \int_0^t \Phi(\theta \cdot s, t - s)B(\theta \cdot s)u(s)ds, \ \ t \in [0, \tau]. \tag{2.2}$$

From now on we will use the parameter $\theta \in \Theta$ only when it is necessary.

Definition 2.1 (Approximate Controllability)**.** The system (1.5) is said to be approximately controllable on $[0, \tau]$ if for all $\theta \in \Theta$ and z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (2.2) corresponding to u verifies

$$\|z(\tau) - z_1\| < \varepsilon.$$

Remark 2.2. Whenever there is no confusion we will omit the variable θ if it is necessary, it is understood that the controllability is uniform in θ variable.

Definition 2.3. For the system (1.5) we define the following concept: The controllability map (for $\tau > 0$) $G = G(\theta) : L^2(0, \tau; U) \to Z$ is given by

$$Gu = \int_0^\tau \Phi(\theta \cdot s, \tau - s)B(\theta \cdot s)u(s)ds,$$

whose adjoint operator $G^* : Z \to L^2(0, \tau; Z)$ is

$$(G^* z)(s) = B^*(\theta \cdot s)\Phi^*(\theta \cdot s, t - s)z, \quad \forall s \in [0, \tau], \quad \forall z \in Z, \tag{2.3}$$

as a consequence of Corollary 1.5.

Proposition 2.4. *For all $C \in L_\infty(0, \tau; L(U, Z))$, the operator $W : L^2(0, \tau; U) \to Z$ given by*

$$W(u) = \int_0^\tau \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds$$

is compact.

Proof. In fact, for $0 < \delta < \tau$, the operator W can be written as

$$W(u) = \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds + \int_{\tau-\delta}^{\tau} \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds.$$

By setting

$$W_\delta(u) = \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds$$

and

$$S_\delta(u) = \int_{\tau-\delta}^{\tau} \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds,$$

we obtain

$$W = W_\delta + S_\delta.$$

Claim 1. The operator W is compact. In fact,

$$\begin{aligned}
W_\delta(u) &= \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - s)C(s)u(s)ds \\
&= \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - \delta - s + \delta)C(s)u(s)ds \\
&= \int_0^{\tau-\delta} \Phi(\theta \cdot s \cdot (\tau - \delta - s), \delta)\Phi(\theta \cdot s, \tau - \delta - s)C(s)u(s)ds \\
&= \int_0^{\tau-\delta} \Phi(\theta \cdot (\tau - \delta - s + s), \delta)\Phi(\theta \cdot s, \tau - \delta - s)C(s)u(s)ds \\
&= \int_0^{\tau-\delta} \Phi(\theta \cdot (\tau - \delta), \delta)\Phi(\theta \cdot s, \tau - \delta - s)C(s)u(s)ds \\
&= \Phi(\theta \cdot (\tau - \delta), \delta) \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - \delta - s)C(s)u(s)ds \\
&= \Phi(\theta \cdot (\tau - \delta), \delta)H_\delta(u).
\end{aligned}$$

where

$$H_\delta(u) = \int_0^{\tau-\delta} \Phi(\theta \cdot s, \tau - \delta - s) C(s) u(s) ds.$$

Since $\Phi(\theta \cdot (\tau - \delta), \delta)$ is compact and the operator H_δ is bounded, then W_δ is compact.

Claim 2. For $\epsilon > 0$, there exists $\delta > 0$ such that $\|S_\delta\| < \epsilon$. In fact, applying Hölder's inequality, we obtain

$$\begin{aligned} \|S_\delta\| &= \left\| \int_{\tau-\delta}^{\delta} \Phi(\theta \cdot s, \tau - s) C(s) u(s) ds \right\| \\ &\leq \int_{\tau-s}^{\tau} \|\Phi(\theta \cdot s, \tau - \delta)\| \, \|C(s)\| \, \|u(s)\| ds \\ &\leq M \|C\|_\infty \int_{\tau-\delta}^{\tau} \|u(s)\| ds \;\leq\; M \|C\|_\infty \delta^{\frac{1}{2}} \|u\|_{L^2}. \end{aligned}$$

By taking

$$\delta^{\frac{1}{2}} \leq \frac{\epsilon}{M \, \|C\|_\infty},$$

we obtain that

$$\|W - W_{\delta_n}\| = \|S_\delta\| < \epsilon.$$

So, we can build a sequence $\{\delta_n\}_{n \geq 1}$ such that it goes to zero and

$$\|W - W_{\delta_n}\| \leq \|S_{\delta_n}\| < \frac{1}{n}, \quad n \, = \, 1, 2 \cdots .$$

Hence, the sequence of compact operators $\{W_{\delta_n}\}$ converges uniformly to W. Therefore, applying results from linear operator theory, we obtain that W is compact. □

Corollary 2.5. *The operator $G(\theta)$ is compact.*

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z.

Lemma 2.6 (See [8, 9, 20])**.** *The equation* (1.5) *is approximately controllable on* $[0, \tau]$ *if and only if for all* $\theta \in \Theta$ *one of the following statements holds:*

a) $\overline{\text{Rang}(G(\theta))} = Z$.

b) $\text{Ker}(G(\theta)^*) = \{0\}$.

c) $\langle G(\theta) G(\theta)^* z, z \rangle > 0$*,* $z \neq 0$ *in* Z.

d) $\lim\limits_{\alpha \to 0^+} \alpha(\alpha I + G(\theta) G(\theta)^*)^{-1} z = 0$.

e) $B^*(\theta \cdot s) \Phi^*(\theta \cdot s, t - s) z = 0$, $\forall t \in [0, \tau]$, $\;\to z = 0$.

f) For all $z \in Z$ we have $G(\theta)u_\alpha = z - \alpha(\alpha I + G(\theta)G(\theta)^*)^{-1}z$, where

$$u_\alpha = G(\theta)^*(\alpha I + G(\theta)G(\theta)^*)^{-1}z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha \to 0} G(\theta)u_\alpha = z$ and the error $E_\alpha z$ of this approximation is given by the formula

$$E_\alpha z = \alpha(\alpha I + G(\theta)G(\theta)^*)^{-1}z, \quad \alpha \in (0,1].$$

Remark 2.7. Lemma 2.6 implies the family of linear operators $\Gamma_\alpha : Z \to L^2(0,\tau;U)$, defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = B^*(\theta \cdot s)\Phi^*(\theta \cdot s, \tau - s)(\alpha I + G(\theta)G(\theta)^*)^{-1}z = G(\theta)^*(\alpha I + G(\theta)G(\theta)^*)^{-1}z,$$

is an approximate inverse for the right of the operator $G(\theta)$, in the sense that

$$\lim_{\alpha \to 0} G(\theta)\Gamma_\alpha = I, \quad \forall \theta \in \Theta,$$

in the strong topology.

Proposition 2.8. *If $\overline{\mathrm{Rang}(G(\theta))} = Z$, then*

$$\sup_{\alpha > 0} \|\alpha(\alpha I + G(\theta)G(\theta)^*)^{-1}\| \leq 1. \tag{2.4}$$

3 Controllability of Nonlinear Systems

In this section we shall prove the main result of this paper, the approximate controllability of the semilinear time varying system (1.1). To this end, we notice that for all $z_0 \in Z$ and $u \in L^2(0,\tau;U)$ the initial value problem

$$\begin{cases} z' = A(\theta \cdot t)z + B(\theta \cdot t)u(t) + F(\theta \cdot t, z, u(t)), \\ z(0) = z_0 \end{cases} \tag{3.1}$$

where $t \in [0,\tau]$ and $z \in Z$, admits only one mild solution given by

$$\begin{aligned} z_u(t) &= \Phi(\theta \cdot s, t-s)z_0 + \int_0^t \Phi(\theta \cdot s, t-s)B(\theta \cdot s)u(s)ds \\ &+ \int_0^t \Phi(\theta \cdot s, t-s)F(\theta \cdot s, z_u(s), u(s))ds, \ t \in [0,\tau]. \end{aligned}$$

Definition 3.1 (Approximate Controllability)**.** The system (1.1) is said to be approximately controllable on $[0,\tau]$ if for every $\theta \in \Theta$ and z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0,\tau;U)$ such that the solution $z(t)$ of (3.2) corresponding to u verifies

$$\|z(\tau) - z_1\| < \varepsilon.$$

Definition 3.2. For the system (1.1) we define the following concept: The nonlinear controllability map (for $\tau > 0$) $G_F = G_F(\theta) : L^2(0,\tau;U) \to Z$ is given by

$$\begin{aligned} G_F u &= \int_0^\tau \Phi(\theta \cdot s, t-s)B(\theta \cdot s)u(s)ds \\ &+ \int_0^\tau \Phi(\theta \cdot s, t-s)F(\theta \cdot s, z_u(s), u(s))ds \\ &= G(u) + H(u), \end{aligned}$$

where $H = H(\theta) : L^2(0,\tau;U) \to Z$ is the nonlinear operator given by

$$H(u) = \int_0^\tau \Phi(\theta \cdot s, t-s)F(\theta \cdot s, z_u(s), u(s))ds, \;\; u \in L^2(0,\tau;U). \tag{3.2}$$

The following lemma is trivial.

Lemma 3.3. *The equation* (1.1) *is approximately controllable on* $[0,\tau]$ *if and only if for all* $\theta \in \Theta$ $\overline{\text{Rang}(G_F(\theta))} = Z$.

Definition 3.4. The following equation will be called the controllability equations associated to the nonlinear equation (1.1)

$$u = \Gamma_\alpha(z - H(u_\alpha)) = G^*(\alpha I + GG^*)^{-1}(z - H(u)), \quad (0 < \alpha \leq 1).$$

Theorem 3.5. *The system* (1.1) *is approximately controllable on* $[0,\tau]$. *Moreover, for all* $\theta \in \Theta$ *a sequence of controls steering the system* (1.1) *from initial state* z_0 *to an* ϵ*-neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) \;=\; B^*(\theta \cdot s)\Phi^*(\theta \cdot s, \tau - s)(\alpha I + GG^*)^{-1}(z_1 - \Phi(\theta,\tau)z_0 - H(u_\alpha)),$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - \Phi(\theta,\tau)z_0 - H(u_\alpha)).$$

Proof. For each $\theta \in \Theta$ and $z \in Z$ fixed, we shall consider the family of nonlinear operators $K_\alpha : L^2(0,\tau;U) \to L^2(0,\tau;U)$ given by

$$K_\alpha(u) = \Gamma_\alpha(z - H(u)) = G^*(\alpha I + GG^*)^{-1}(z - H(u)), \quad (0 < \alpha \leq 1).$$

First, we shall prove that for all $\alpha \in (0,1]$ the operator K_α has a fixed point u_α. In fact, since F is smooth and satisfies (1.4) and the evolution operator $\Phi(\theta,t)$ given by (1.2) is compact, then using the ideas of the proof of Proposition 2.5 and (1.4) we can prove that the operator H is compact. Moreover,

$$\overline{\lim}_{\|u\|_{L_2}\to\infty} \frac{\|K_\alpha(u)\|_{L_2}}{\|u\|_{L_2}} = 0. \tag{3.3}$$

In fact, from the definition of the operator $H(u)$, Propositions 1.2 and 1.3 we have, for $u \in L^2(0,\tau;U)$, the estimate

$$\begin{aligned}
\|H(u)\| &\leq \int_0^\tau Me^{\omega(\tau-s)}\|F(s,z_u(s),u(s))\|ds \\
&\leq \left(\int_0^\tau M^2e^{2\omega(\tau-s)}ds\right)^{1/2}\left(\int_0^\tau \|F(s,z_u(s),u(s))\|^2ds\right)^{1/2} \\
&= N\left(\int_0^\tau \|F(s,z_u(s),u(s))\|^2ds\right)^{1/2} \\
&\leq N\left(\int_0^\tau \left(a\|u(s)\|^\beta+b\right)^2ds\right)^{1/2} \\
&\leq N\left(\int_0^\tau (4a^2\|u(s)\|^{2\beta}+4b^2)ds\right)^{1/2} \\
&\leq 2Na\left(\int_0^\tau \|u(s)\|^{2\beta}\right)^{1/2}+2b\sqrt{\tau} \\
&\leq 2Na\left\{\left(\int_0^\tau \|u(s)\|^{2\beta}\right)^{1/(2\beta)}\right\}^{\beta}+2b\sqrt{\tau} \\
&= 2aN\|u\|^\beta_{L_{2\beta}}+2b\sqrt{\tau}.
\end{aligned}$$

Now, since $1/2 \leq \beta < 1$ iff $1 \leq 2\beta < 2$, applying Proposition 1.3, we obtain that:

$$\|H(u)\| \leq 2aN\tau^{\frac{1-\beta}{2\beta}}\|u\|^\beta_{L_2}+2b\sqrt{\tau}.$$

Therefore,

$$\overline{\lim}_{\|u\|_{L_2}\to\infty}\frac{\|H(u)\|_{L_2}}{\|u\|_{L_2}}=0.$$

Consequently,

$$\overline{\lim}_{\|u\|_{L_2}\to\infty}\frac{\|K_\alpha(u)\|}{\|u\|_{L_2}}=0.$$

Then, from condition (3.3) we obtain that, for a fixed $0<\rho<1$, there exists R_α such that

$$\|K_\alpha(u)\|_{L_2} \leq \rho\|u\|_{L_2}, \ \ \|u\|_{L_2}=R_\alpha.$$

Hence, if we denote by $B(0,R_\alpha)$ the ball of center zero and radio $R_\alpha>0$, we get that $K_\alpha(\partial B(0,R_\alpha)) \subset B(0,R_\alpha)$. Since K_α is compact and maps the sphere of $\partial B(0,R_\alpha)$ into the interior of the ball $B(0,R_\alpha)$, we can apply Rothe's fixed point theorem (Theorem 1.7) to ensure the existence of a control $u_\alpha \in L_2(0,\tau;U)$ such that

$$u_\alpha = K_\alpha u_\alpha = \Gamma_\alpha(z-H(u_\alpha)) = G^*(\alpha I+GG^*)^{-1}(z-H(u_\alpha)), \quad (0<\alpha\leq 1).$$

Claim. The family of fixed pint $\{u_\alpha\}_{0<\alpha\leq 1}$ is bounded. In fact, for the purpose of contradiction, let us assume the contrary. Then, there exists a subsequence $\{u_{\alpha_n}\}_{n\geq 1} \subset \{u_\alpha\}_{0<\alpha\leq 1}$ such that

$$\lim_{n\to\infty} \|u_{\alpha_n}\|_{L_2} = \infty.$$

On the other hand, from (3.3) we get for all fixed $\alpha \in (0,1]$ that

$$\overline{\lim}_{n\to\infty} \frac{\|K_\alpha(u_{\alpha_n})\|}{\|u_{\alpha_n}\|_Z} = 0.$$

Hence,

$$\begin{array}{cccccc}
\frac{\|K_{\alpha_1}(u_{\alpha_1})\|}{\|u_{\alpha_1}\|_Z} & \frac{\|K_{\alpha_1}(u_{\alpha_2})\|}{\|u_{\alpha_2}\|_Z} & \frac{\|K_{\alpha_1}(u_{\alpha_3})\|}{\|u_{\alpha_3}\|_Z} & \dots & \frac{\|K_{\alpha_1}(u_{\alpha_n})\|}{\|u_{\alpha_n}\|_Z} & \to 0 \\
\frac{\|K_{\alpha_2}(u_{\alpha_1})\|}{\|u_{\alpha_1}\|_Z} & \frac{\|K_{\alpha_2}(u_{\alpha_2})\|}{\|u_{\alpha_2}\|_Z} & \frac{\|K_{\alpha_2}(u_{\alpha_3})\|}{\|u_{\alpha_3}\|_Z} & \dots & \frac{\|K_{\alpha_2}(u_{\alpha_n})\|}{\|u_{\alpha_n}\|_Z} & \to 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\|K_{\alpha_n}(u_{\alpha_1})\|}{\|u_{\alpha_1}\|_Z} & \frac{\|K_{\alpha_n}(u_{\alpha_2})\|}{\|u_{\alpha_2}\|_Z} & \frac{\|K_{\alpha_n}(u_{\alpha_3})\|}{\|u_{\alpha_3}\|_Z} & \dots & \frac{\|K_{\alpha_n}(u_{\alpha_n})\|}{\|u_{\alpha_n}\|_Z} & \to 0.
\end{array}$$

Now, applying Cantor's diagonalization process, we obtain that

$$\overline{\lim}_{n\to\infty} \frac{\|K_{\alpha_n}(u_{\alpha_n})\|}{\|u_{\alpha_n}\|_Z} = 0.$$

From the fixed point property, we have that $u_{\alpha_n} = K_{\alpha_n} u_{\alpha_n}$. So,

$$\|u_{\alpha_n}\|_{L_2} = \|K_{\alpha_n} u_{\alpha_n}\|_{L_2} \iff \frac{\|K_{\alpha_n}(u_{\alpha_n})\|_{L_2}}{\|u_{\alpha_n}\|_{L_2}} = 1.$$

Hence,

$$\overline{\lim}_{\|u_{\alpha_n}\|_{L_2}\to\infty} \frac{\|K_{\alpha_n}(u_{\alpha_n})\|_{L_2}}{\|u_{\alpha_n}\|_{L_2}} = 1,$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma > 0$ such that

$$\|u_{\alpha_n}\|_{L_2} \leq \gamma, \ \ (0 < \alpha \leq 1).$$

Therefore, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha(z - H(u)) = G^*(\alpha I + GG^*)^{-1}(z - H(u_\alpha)),$$

then

$$\begin{aligned} Gu_\alpha &= G\Gamma_\alpha(z - H(u)) = GG^*(\alpha I + GG^*)^{-1}(z - H(u_\alpha)) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z - H(u_\alpha)) \\ &= z - H(u_\alpha) - \alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha)). \end{aligned}$$

Hence,

$$Gu_\alpha + H(u_\alpha) = z - \alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha)).$$

To conclude the proof, it enough to prove that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha))\} = 0.$$

From Lemma 2.6.d) we get that

$$\begin{aligned} \lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha))\} &= -\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}H(u_\alpha)\} \\ = -\lim_{\alpha\to 0} -\alpha(\alpha I + GG^*)^{-1}y &- \lim_{\alpha\to 0} -\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y) \\ &= \lim_{\alpha\to 0} -\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y). \end{aligned}$$

On the other hand, from Proposition 2.8, we get that

$$\|\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y)\| \le \|(H(u_\alpha) - y)\|.$$

Therefore, since $H(u_\alpha)$ converges to y, we get that

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}(H(u_\alpha) - y)\} = 0.$$

Consequently,

$$\lim_{\alpha\to 0}\{-\alpha(\alpha I + GG^*)^{-1}(z - H(u_\alpha))\} = 0.$$

So, putting $z = z_1 - \Phi(\theta,\tau)z_0$ and using (3.2), we obtain the nice result:

$$\begin{aligned} z_1 &= \lim_{\alpha\to 0^+}\left\{\Phi(\theta,\tau)z_0 + \int_0^\tau \Phi(\theta\cdot s, \tau - s)B(\theta\cdot s)u_\alpha(s)ds\right. \\ &+ \int_0^\tau \Phi(\theta\cdot s, \tau - s)F(s, z_{u_\alpha}(s), u_\alpha(s))ds\}. \end{aligned}$$

This completes the proof. □

4 Applications

In this section we use the foregoing results to prove the interior approximate controllability of the following broad class of nonautonomous reaction diffusion equation in the Hilbert space $Z = L^2(\Omega)$ given by

$$\begin{cases} z' = A(\theta \cdot t)z + B(\theta \cdot t)u(t) + F(\theta \cdot t, z, u(t)), & t \in [0, \tau], \quad \theta \in \Theta \\ z(0) = z_0 \end{cases} \tag{4.1}$$

where the control $u \in L^2(0, \tau; L^2(\Omega))$, with $U = Z$, Θ is a compact Hausdorff space, $A(\theta \cdot t) = -a(\theta \cdot t)A$ with $a : \Theta \to \mathbb{R}_+$ a continuous function, the operator $B : Z \to Z$ is a linear and bounded and $A : D(A) \subset Z \to Z$ is an unbounded linear operator with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{4.2}$$

with $\langle \cdot, \cdot \rangle$ denoting an inner product in Z, and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}.$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A. So, $\{E_j\}$ is a complete family of orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z$, $z \in Z$. The nonlinear function $F : \Theta \times Z \times U \to Z$ is smooth enough and there are $a, b \in \mathbb{R}, R > 0$ and $1/2 \leq \beta < 1$ such that

$$\|F(\theta, z, u)\|_Z \leq a\|u\|^{\beta} + b, \ \ \forall u, z \in Z, \theta \in \Theta. \tag{4.3}$$

The operator $-A(\theta \cdot t)$ generates a compact evolution operator $\Phi(\theta \cdot t)$ given by

$$\Phi(\theta \cdot t)z = T(g_\theta(t))z = \sum_{j=1}^{\infty} e^{-\lambda_j g_\theta(t)} E_j z,$$

where $g_\theta(t) = \int_0^t a(\theta \cdot s)ds > 0, t \neq 0$.

Theorem 4.1. *If for all $\theta \in \Theta$ the vectors $B^*(\theta)\phi_{j,k}$ are linearly independent in Z, then the system* (4.1) *is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system* (4.1) *from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B^*(\theta \cdot t)\Phi^*(\theta \cdot t, \tau - t)(\alpha I + GG^*)^{-1}(z_1 - \Phi(\theta, \tau)z_0 - H(u_\alpha)),$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - \Phi(\theta,\tau)z_0 - H(u_\alpha)),$$

where

$$H(u) = \int_0^\tau \Phi(\theta\cdot s, \tau - s)F(\theta\cdot s, z_u(s), u(s))ds, \;\; u \in L^2(0,\tau;U).$$

Proof. It is enough to prove that the linear part of this system (4.1)

$$\begin{cases} z' = A(\theta\cdot t)z + B(\theta\cdot t)u(t), \;\; t\in[0,\tau], \quad \theta\in\Theta \\ z(0) = z_0 \end{cases} \tag{4.4}$$

is approximately controllable on $[0,\tau]$. To this end, we shall apply Lemma 2.6.e). We observe that

$$\Phi^\star(\theta,t)z \;=\; T^*(g(\theta\cdot t))z \;=\; \sum_{j=1}^\infty e^{-\lambda_j g(\theta\cdot t)}E_j z,$$

where $g(\theta\cdot t) = \displaystyle\int_0^t a(\theta\cdot s)ds \neq 0$. Then,

$$\begin{aligned} B^*(\theta\cdot t)\Phi^*(\theta,t)z &= B^*(\theta\cdot t)\sum_{j=1}^\infty e^{-\lambda_j g(\theta\cdot t)}E_j z \\ &= \sum_{j=1}^\infty e^{-\lambda_j g(\theta\cdot t)}B^*(\theta\cdot t)E_j z = 0, \;\; \forall\, t\in[0,\tau]. \end{aligned}$$

Therefore, from Lemma 1.6 we obtain that

$$B^*(\theta\cdot t)E_j z = \sum_{k=1}^{\gamma_j}\langle z,\phi_{j,k}\rangle B^*(\theta\cdot t)\phi_{j,k} = 0, \quad j = 1,2,\ldots.$$

Since, by hypothesis, the set of all vectors $B^*(\theta\cdot t)\phi_{j,k}$ are linearly independents, we obtain that

$$\langle z,\phi_{j,k}\rangle = 0 \quad j = 1,2,\ldots. \quad \text{and} \quad k = 1,2,\ldots,\gamma_j.$$

So, $E_j z = 0, \quad j = 1,2,\ldots,$ which implies that $z = 0$ Hence, the system (4.4) is approximately controllable on $[0,\tau]$. □

Example 4.2 (The Interior Controllability of the nD Time-Varying Heat Equation)**.** In this case, as an example, we prove the interior approximate controllability of the semilinear time-varying heat equation

$$\begin{cases} z_t(t,x) = a(\theta\cdot t)\Delta z(t,x) + 1_\omega u(t,x) + f(\theta\cdot t, z, u(t,x)) & \text{in} \quad (0,\tau]\times\Omega, \\ z = 0, & \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(0,x) = z_0(x), & x\in\Omega, \end{cases} \tag{4.5}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $L^2([0,\tau]; L^2(\Omega;))$, with $a : \Theta \to \mathbb{R}_+$ and the nonlinear function $f : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth enough and there are $a, b \in \mathbb{R}$ and $1/2 \leq \beta < 1$ such that

$$|f(\theta, z, u)| \leq a|u|^\beta + b, \ \ \forall u, z \in \mathbb{R}, \theta \in \Theta. \tag{4.6}$$

In this case $B(\theta \cdot t)f = 1_\omega f$.

To this end, we shall describe the space in which this problem will be situated as an abstract ordinary differential equation. Let us consider $Z = L^2(\Omega)$ and the linear unbounded operator $A : D(A) \subset Z \to Z$ defined by $A(\theta \cdot t)\phi = -a(\theta \cdot t)\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{4.7}$$

The operator A has the following very well known properties: The spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty,$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of A.

b) For all $z \in D(A)$ we have

$$-\Delta z = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{4.8}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X and

$$E_n z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \tag{4.9}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \ \ z \in Z. \tag{4.10}$$

c) $-a(\theta)\Delta$ generates a compact evolution operator $\Phi(\theta \cdot t)$ given by

$$\Phi(\theta \cdot t)z = T(g_\theta(t))z = \sum_{j=1}^{\infty} e^{-\lambda_j g_\theta(t)} E_j z,$$

where $g_\theta(t) = \displaystyle\int_0^t a(\theta \cdot s)ds \neq 0$.

The system (4.5) can be written as an abstract equation in the space $Z = L^2(\Omega)$

$$\begin{cases} z' = -A(\theta \cdot t)z + B(\theta \cdot t)u(t) + F(\theta \cdot t, z, u), z \in Z, \\ z(0) = z_0, \end{cases} \tag{4.11}$$

where the control function u belongs to $L^2(0, \tau; Z)$ and $F : \Theta \times Z \times U \to Z$, is defined by $F(\theta, z, u)(x) = f(\theta, z(x), u(x)), \ \forall x \in \Omega$.

Proposition 4.3. *The vectors $B^*(\theta)\phi_{j,k} = 1_\omega \phi_{j,k}$ are linearly independent in Z.*

Proof. Consider any linear combination of the form

$$\sum_{k=1}^{\gamma_j} a_{j,k} B^*(\theta \cdot t)\phi_{j,k}(z) = \sum_{k=1}^{\gamma_j} a_{j,k} 1_\omega \phi_{j,k}(z) = 0, \quad \forall z \in \Omega.$$

Therefore,

$$\sum_{k=1}^{\gamma_j} a_{j,k} 1_\omega \phi_{j,k}(z) = \sum_{k=1}^{\gamma_j} a_{j,k} \phi_{j,k}(z) = 0, \quad \forall z \in \omega.$$

Now, putting $g(z) = \sum_{k=1}^{\gamma_j} a_{j,k}\phi_{j,k}(z)$, $z \in \Omega$, we obtain that

$$\begin{cases} (\Delta + \lambda_j I)g \equiv 0 \quad \text{in} \quad \Omega, \\ g(x) = 0 \quad \forall x \in \omega. \end{cases}$$

Then, from the classical unique continuation for elliptic equations (see [27]), it follows that $g(z) = 0$, $z \in \Omega$. So,

$$\sum_{k=1}^{\gamma_j} a_{j,k}\phi_{j,k}(x) = 0, \quad \forall x \in \Omega.$$

On the other hand, $\{\phi_{j,k}\}$is a complete orthonormal set in $Z = L^2(\Omega)$, which implies that $a_{j,k} = 0$. $\square$

Proposition 4.4. *Under the condition* (4.6)*, the function $F : \theta \times Z \times U \to Z$ defined by $F(\theta \cdot t, z, u)(x) = f(\theta \cdot t, z(x), u(x))$, $x \in \Omega$, satisfies for all $u, z \in Z = L^2(\Omega), \theta \in \Theta$:*

$$\|F(\theta, z, u)\|_Z \le 2a\mu(\Omega)^{(1-\beta)/2}\|u\|_Z^\beta + 2b\sqrt{\mu(\Omega)}. \tag{4.12}$$

Proof. We have

$$\begin{aligned}
\|F(\theta,z,u)\|_Z &= \left(\int_\Omega |f(\theta,z(x),u(x))|^2 dx\right)^{1/2} \\
&\le \left(\int_\Omega \left(a|u(x)|^\beta + b\right)^2 dx\right)^{1/2} \\
&\le \left(\int_\Omega (4a^2\|u(x)\|^{2\beta} + 4b^2) dx\right)^{1/2} \\
&\le 2a\left\{\left(\int_\Omega \|u(x)\|^{2\beta}\right)^{\frac{1}{2\beta}}\right\}^\beta + 2b\sqrt{\mu(\Omega)} \\
&= 2a\|u\|^\beta_{L_{2\beta}(\Omega)} + 2b\sqrt{\mu(\Omega)}.
\end{aligned}$$

Now, since $1/2 \le \beta < 1$ iff $1 \le 2\beta < 2$, we obtain

$$\|F(\theta,z,u)\|_Z \le 2a\mu(\Omega)^{\frac{1-\beta}{\beta}}\|u\|^\beta_Z + 2b\sqrt{\mu(\Omega)}$$

by applying Proposition 1.3. □

Using the compactness of the evolution operator

$$\Phi(\theta\cdot t)z = T(g_\theta(t))z = \sum_{j=1}^\infty e^{-\lambda_j g_\theta(t)} E_j z$$

generated by $-a(\theta)\Delta$, where $g_\theta(t) = \displaystyle\int_0^t a(\theta\cdot s)ds > 0$, and the foregoing two propositions, we can prove that the semilinear heat equation (4.5) is approximately controllable on $[0,\tau]$.

Theorem 4.5. *The system* (4.5) *is approximately controllable on* $[0,\tau]$. *Moreover, a sequence of controls steering the system* (4.5) *from initial state* z_0 *to an* ϵ *neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = 1_\omega T(g_\theta(\tau-t))(\alpha I + GG^*)^{-1}(z_1 - T(g_\theta(t))z_0 - H(u_\alpha)),$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(g_\theta(\tau))z_0 - H(u_\alpha)),$$

where

$$H(u) = \int_0^\tau T(g_\theta(\tau-t))F(\theta\cdot s, z_u(s), u(s))ds, \;\; u \in L^2(0,\tau;U)$$

Acknowledgements

This work has been supported by CDCHT-ULA-C-1796-12-05-AA and BCV.

References

[1] Arias De Reyna J., Diestel J., Lomonosov V. and Rodriguez Piazza L., Some observations about the space of weakly continuous functions from a compact space into a Banach space, Quaest. Math., 415–425 (1994).

[2] D. Barcenas, S.N. Chow, H. Leiva and A. Tineo Moya, Skew-product semi-flows and nonautonomous control systems, J. Math. Anal. Appl. 381 247–262 (2011).

[3] H. Brezis, *Analisis Funcional, Teoria y Applicaciones*, Alianza Universitaria Textos, Masson, Paris, 1983. Ed. cast.: Alinza Editorial, S. A., Madrid, 1984.

[4] S.N. Chow and H. Leiva, Dynamical spectrum for time dependent linear systems in Banach spaces, Japan J. Indust. Appl. Math., 11, 379–415 (1994).

[5] S.N. Chow and H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces, J. Diff. Eqs., 120., 429–477 (1995).

[6] S.N. Chow and H. Leiva, Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces, Proc. Amer. Math. Soc., 124, 1071–1081 (1996).

[7] S.N. Chow and H. Leiva, Dynamical spectrum for skew-product semiflow in Banach spaces, boundary problems for functional differential equations, World Sci. Publ., Singapore, 85–105 (1995).

[8] R.F. Curtain and A.J. Pritchard, *Infinite Dimensional Systems, Lecture Notes in Control and Information Sciences*, Vol. 8. Springer Verlag, Berlin (1978).

[9] R.F. Curtain and H.J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Text in Applied Mathematics, **21**. Springer Verlag, New York (1995).

[10] J.P. Daur and N.I. Mahmudov, Approximate controllability of some semilinear functional equations in Hilbert spaces, J. Math. Anal. 273(2), 310–327 (2002).

[11] J.P. Daur and N.I. Mahmudov, Controllability of some nonlinear systems in Hilbert spaces, J. Optim. Theory Appl. 123(2), 319–329 (2004).

[12] J.I. Diaz, J. Henry and A.M. Ramos, On the approximate controllability of some semilinear parabolic boundary-value problems, Appl. Math. Optim. 37–71 (1998).

[13] E. Fernadez-Cara, Remark on approximate and null controllability of semilinear parabolic equations, ESAIM: Proceedings of Controle et Equations aux Derivees Partielles, Vol. 4, 73–81 (1998).

[14] E. Fernandez-Cara and E. Zuazua, Controllability for blowing up semilinear parabolic equations, C.R. Acad. Sci. Paris, t. 330, serie I, p. 199–204 (2000).

[15] D. Hinrichsen and A.J. Pritchard, Robust stability of linear evolution operators on Banach spaces, report No. 269 (1992), Institut für Dynamische Systeme, F.M.I Universität Bremen, Germany.

[16] G. Isac, On Rothe's fixed point theorem in general topological vector space, An. St. Univ. Ovidius Constanta, Vol. 12(2), 127–134 (2004).

[17] R. Johnson and M. Nerurkar, On null controllability of linear systems with recurrent coefficients and constrained controls, J. Dynamics Differential Eq., 4(2), 1992.

[18] H. Leiva, N. Merentes and J.L. Sanchez, Interior controllability of the nD semilinear heat equation, Special volume in Honor of Profs. C. Corduneanu, A. Fink, and S. Zaidman, African Diasp. J. Math., 12(2), 1–12 (2011).

[19] H. Leiva, Approximate controllability of semilinear cascade systems in $H = L^2(\Omega)$, Int. Math. Forum, 7(57), 2797–2813 (2012).

[20] H. Leiva and Y. Quintana, Interior controllability of a broad class of reaction diffusion equation, Math. Problems Engineering, Vol. 2009, Article ID 708516, 8 pages.

[21] V.I. Korobov and R. Rabakh, Exact controllability in Banach spaces, A.M. Gorki Kharkov State University. Translated from Differentsial'nye Uravheniya, 5(12), 2142–2150 (1979).

[22] E.B. Lee and Markus, *Foundations of Optimal Control Theory*, John Wiley, New York (1967).

[23] M. Megan, A.L. Sasu and B. Sasu, On approximate controllability Of systems associated to linear skew-product semiflows, Analele Stiintifice ale Universitătii "AL.I. Cuza" IASI Tomul XLVII, s.I a, Matematică , 2001, f.2.

[24] K. Naito, Controllability of semilinear control systems dominated by the linear part, SIAM J. Control Optim. 25(3), (1987).

[25] K. Naito, Approximate controllability for trajectories of semilinear control systems, J. Optimization Th. Appl., 60(1), (1989).

[26] Nazim I. Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, SIAM J. Control Optim., 42(5), 1604–1622.

[27] M.H. Protter, Unique continuation for elliptic equations, Trans. Amer. Math. Soc., 95(1), 1960.

[28] L. De Teresa, Approximate controllability of semilinear heat equation in $\mathbb{R}^N$, SIAM J. Control Optim., 36(6), 2128–2147 (1998).

[29] L. De Teresa and E. Zuazua, Approximate controllability of semilinear heat equation in unbounded domains, Nonlinear Anal. 8 (1999).

[30] A.L. Sasu, Stability and controllability for systems of difference equations, J. Differ. Equations Appl., 12, 821–826 (2006).

[31] A.L. Sasu, On exact controllability of variational discrete systems, to appear in Appl. Math. Letters.