Existence of Nontrivial Solutions for a Class of Mixed Boundary Laplacian Systems

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Abstract

In this paper, we prove the existence of nontrivial solutions to a mixed of boundary \((p, q)\)-Laplacian systems. Our technical approach is based on Bonanno’s general critical points theorem.

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1 Introduction and Preliminaries

In this paper, we are concerned with the existence of nontrivial weak solutions for the following system of \((p, q)\)-Laplacian type

\[
\begin{align*}
-\left(|u'|^{p-2}u'\right)' &= \lambda F_u(x, u, v) \quad \text{in} \ (0, 1), \\
-\left(|v'|^{q-2}v'\right)' &= \lambda F_v(x, u, v) \quad \text{in} \ (0, 1), \\
u(0) = u'(1) = v(0) = v'(1) &= 0,
\end{align*}
\]

where \(q \geq p > 1\) and \(\lambda\) is a real positive parameter. \(F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}\) is a function such that \(F(\cdot, s, t)\) is continuous in \([0, 1]\) for all \((s, t) \in \mathbb{R}^2\) and \(F(x, \cdot, \cdot)\) is \(C^1\) in \(\mathbb{R}^2\).
for every $x \in [0, 1]$, and $F_u$, $F_v$ denote the partial derivatives of $F$ with respect to the second and third variable. Moreover, $F(\cdot, s, t)$ satisfies the condition

$$ \sup_{|s| \leq \sigma, |t| \leq \sigma} (|F_u(\cdot, s, t)| + |F_v(\cdot, s, t)|) \in L^1([0, 1]), \text{ for all } \sigma > 0. $$

The investigation of existence of solutions for a mixed boundary value problem has drawn the attention of many authors, see [7, 8, 10, 15, 17] and references therein. In [1], the authors established the existence of infinitely many solutions for the following mixed boundary value problem involving the one dimensional $p$-Laplacian

$$ - (p-2) u'^{p-2} u' + s |u|^{p-2} u = \lambda f(x, u) \quad \text{in} \ (a, b), $$

$$ u(a) = u'(b) = 0, $$

(1.2)

where the functions $r$, $s$ and $f$ satisfy suitable hypotheses. In [2], Averna and Salvati proved that, in the case $r = s = 1$, the problem (1.2) admits at least three solutions when $\lambda$ lies in an exactly determined open interval. Recently, D’Agui in [9], obtained existence results of three nonnegative and nontrivial solutions for problem (1.2).

Motivated by the results mentioned above, in this paper a precise interval of real parameters $\lambda$ for which the problem (1.1) admits at least one nontrivial solutions is established. Our approach is variational and the main tool is a local minimum theorem due to Bonanno [3].

Denote by $X$ the Cartesian product of two Sobolev spaces

$$ X_1 = \{ u \in W^{1,p}([0, 1]); u(0) = 0 \} \quad \text{and} \quad X_2 = \{ v \in W^{1,q}([0, 1]); v(0) = 0 \}. $$

The space $X$ will be endowed with the norm

$$ \|(u, v)\| = ||u||_p + ||v||_q, $$

where

$$ ||u||_p = \left( \int_0^1 |u'(x)|^p dx \right)^{\frac{1}{p}}, \quad \text{and} \quad ||v||_q = \left( \int_0^1 |v'(x)|^q dx \right)^{\frac{1}{q}}. $$

Since $p > 1$ and $q > 1$, the Rellich Kondrachov theorem assures that $(X_1, || \cdot ||_p) \hookrightarrow (C^0([0, 1]), || \cdot ||_\infty)$ and $(X_2, || \cdot ||_q) \hookrightarrow (C^0([0, 1]), || \cdot ||_\infty)$ are compact, therefore

$$ ||u||_\infty < ||u||_p \quad \text{and} \quad ||v||_\infty < ||v||_q \quad \text{for all} \ (u, v) \in X \ (\text{see, e.g.,} \ [16]). $$

(1.3)

**Definition 1.1.** We say that $(u, v) \in X$ is a weak solution of problem (1.1) if

$$ \int_0^1 |u'|^{p-2} u' \varphi' dx + \int_0^1 |v'|^{q-2} v' \psi' dx - \lambda \int_0^1 F_u(x, u, v) \varphi dx $$

$$ - \lambda \int_0^1 F_v(x, u, v) \psi dx = 0, $$

for all $(\varphi, \psi) \in X$. 
We see that weak solutions of system (1.1) are critical points of the functional $I_\lambda : X \to \mathbb{R}$, given by

$$I_\lambda(u, v) = \Phi(u, v) - \lambda \Psi(u, v),$$

for all $(u, v) \in X$, where

$$\Phi(u, v) = \frac{1}{p}||u||_p^p + \frac{1}{q}||v||_q^q \quad \text{and} \quad \Psi(u, v) = \int_0^1 F(x, u, v)dx.$$ 

Since $X$ is compactly embedded in $C^0([0, 1]) \times C^0([0, 1])$, it is well known that $\Phi$ and $\Psi$ are well defined and Gâteaux differentiable functionals whose Gâteaux derivatives at $(u, v) \in X$ are given by

$$\langle \Phi'(u, v), (\varphi, \psi) \rangle = \int_0^1 |u'(x)|^{p-2} u'(x) \varphi'(x) dx + \int_0^1 |v'(x)|^{q-2} v'(x) \psi'(x) dx,$$

$$\langle \Psi'(u, v), (\varphi, \psi) \rangle = \int_0^1 F_u(x, u, v) \varphi dx + \int_0^1 F_v(x, u, v) \psi dx,$$

for all $(\varphi, \psi) \in X$. Moreover, by the weakly lower semicontinuity of norm, we see that $\Phi$ is sequentially weak lower semicontinuous. Thanks to $p, q > 1$ and $(F)$, $\Psi$ has a compact derivative, it follows that $\Psi$ is sequentially weakly continuous.

Our main tools are two consequences of a local minimum theorem [3], which is a more general version of the Ricceri variational principle [14]. Given a set $X$ and two functionals $\Phi, \Psi : X \to \mathbb{R}$, put

$$\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(]-r_2, r_1[)} \frac{\left( \sup_{v \in \Phi^{-1}(]-r_2, r_1[)} \Psi(v) \right) - \Psi(u)}{r_2 - \Phi(u)} \quad (1.4)$$

$$\rho_1(r_1, r_2) := \sup_{u \in \Phi^{-1}(]-r_2, r_1[)} \frac{\Psi(u) - \left( \sup_{v \in \Phi^{-1}(]-r_2, r_1[)} \Psi(v) \right)}{\Phi(u) - r_1} \quad (1.5)$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and

$$\rho(r) := \sup_{u \in \Phi^{-1}(]-\infty, r])} \frac{\Psi(u) - \left( \sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) \right)}{\Phi(u) - r} \quad (1.6)$$

for all $r \in \mathbb{R}$.

**Theorem 1.2** (See [3, Theorem 5.1]). Let $X$ be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse function on $X^*$, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda \Psi$. Assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that

$$\beta(r_1, r_2) < \rho_1(r_1, r_2), \quad (1.7)$$
where $\beta, \rho_1$ are given by (1.4) and (1.5). Then, for each $\lambda \in \left[\frac{1}{\rho_1(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\right]$, there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

**Theorem 1.3** (See [3, Theorem 5.3]). Let $X$ be a real Banach space, $\Phi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse function on $X^*$, $\Psi : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that

$$\rho(r) > 0, \quad (1.8)$$

where $\rho$ is given by (1.6), for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda = \Phi - \lambda \Psi$ is coercive.

Then, for $\lambda > \frac{1}{\rho(r)}$, there is $u_{0,\lambda} \in \Phi^{-1}(r, +\infty)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(r, +\infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

## 2 Main Results

Given two nonnegative constants $c$ and $d$ such that

$$\frac{c^p}{q2^p} \neq \frac{2^q}{p}d^q,$$

put

$$A_d(c) := \frac{\int_0^1 \max_{|s|+|t| \leq c} F(x, s, t)dx - \int_1^\frac{1}{2} F(x, d, d)dx}{\frac{c^p}{q2^p} - \frac{2^q}{p}d^q}.$$  

Now we state our main results.

**Theorem 2.1.** Assume that

(i) there exist three constants $c_1, c_2,$ and $d$ with

$$\max\left(4, 2q^{1/p}\right) \leq c_1 < c_2, \quad (2.1)$$

and

$$\frac{c_1}{4} < d < \left(\frac{pc_2^p}{q2^p+q}\right)^{\frac{1}{q}}, \quad (2.2)$$

such that

$$A_d(c_2) \leq A_d(c_1),$$

(ii) $\int_0^\frac{1}{2} F(x, s, t)dx \geq 0$ for every $(s, t) \in [0, d]^2$. 


Then, for each
\[ \lambda \in \left[ \frac{1}{A_d(c_1)}, \frac{1}{A_d(c_2)} \right], \]
problem (1.1) admits at least one nontrivial weak solution \((\tilde{u}, \tilde{v})\) such that
\[ \frac{c_1^p}{q2^p} < \Phi(\tilde{u}, \tilde{v}) < \frac{c_2^p}{q2^p}. \]

**Proof.** Let \( \Phi, \Psi \) be the functionals defined in Section 1. It is well known that they satisfy all regularity assumptions requested in Theorem 1.2. So, our aim is to verify condition (1.7). Define the function \( u_0 \in X \) by
\[ u_0(x) = \begin{cases} 
2dx, & x \in \left[ 0, \frac{1}{2} \right], \\
d, & x \in \left[ \frac{1}{2}, 1 \right]. 
\end{cases} \]

Clearly \((u_0, u_0) \in X\), and owing to (ii), one has
\[ \Psi(u_0, u_0) = \int_0^1 F(x, u_0, u_0)dx \geq \int_{\frac{1}{2}}^1 F(x, d, d)dx. \]  
(2.3)

From the definition of \( \Phi \) and the conditions (2.1) and (2.2), we have
\[ \frac{2^p}{q}d^p \leq \Phi(u_0, u_0) \leq \frac{2^q}{p}d^q. \]  
(2.4)

Fix \( c_1, c_2, \) and \( d \) satisfying (i) and put
\[ r_1 = \frac{c_1^p}{q2^p} \quad \text{and} \quad r_2 = \frac{c_2^p}{q2^p}. \]

Then, by (2.2), we obtain \( r_1 < \Phi(u_0, u_0) < r_2 \). Moreover, from (1.3) and (2.1) for all \((u, v) \in X\) such that \((u, v) \in \Phi^{-1}(\{-\infty, r_2\})\), we get
\[ |u| + |v| \leq c_2. \]

Therefore
\[ \Psi(u, v) = \int_0^1 F(x, u, v)dx \leq \int_0^1 \max_{|s|+|t|\leq c_2} F(x, s, t)dx \quad \text{for all } (u, v) \in \Phi^{-1}(\{-\infty, r_2\}). \]

Hence
\[ \sup_{(u, v) \in \Phi^{-1}(\{-\infty, r_2\})} \Psi(u, v) \leq \int_0^1 \max_{|s|+|t|\leq c_2} F(x, s, t)dx. \]  
(2.5)
By the same argument, we obtain

\[ \sup_{(u,v) \in \Phi^{-1}([-\infty, r_1])} \Psi(u,v) \leq \int_0^1 \max_{|s|+|t| \leq c_1} F(x, s, t)dx. \tag{2.6} \]

Combining (2.3), (2.4), (2.5) and (2.6), we get

\[
\beta(r_1, r_2) \leq \left( \sup_{(u,v) \in \Phi^{-1}([r_1, r_2])} \Psi(u,v) - \Psi(u_0, u_0) \right) \\
\leq \int_0^1 \max_{|s|+|t| \leq c_2} F(x, s, t)dx - \int_0^1 F(x, d, d)dx \\
\frac{c_2^p}{q2^p} - \frac{2^p}{q} d^q = A_d(c_2),
\]

and

\[
\rho_1(r_1, r_2) \geq \frac{\Psi(u_0, u_0) - \left( \sup_{v \in \Phi^{-1}([-\infty, r_1])} \Psi(v) \right) - \Phi(u_0, u_0)}{r_1} \\
\geq \frac{\int_0^1 F(x, d, d)dx - \int_0^1 \max_{|s|+|t| \leq c_2} F(x, s, t)dx}{\frac{2^p}{q} d^q - \frac{c_2^p}{q2^p}} = A_d(c_1).
\]

So, by (i) it follows that

\[ \beta(r_1, r_2) < \rho_1(r_1, r_2). \]

Hence, by Theorem 1.2 for each \( \lambda \in \left[ \frac{1}{A_d(c_1)}, \frac{1}{A_d(c_2)} \right] \), the functional \( I_\lambda \) admits at least one critical point \((\tilde{u}, \tilde{v})\) such that

\[ \frac{c_1^p}{q2^p} < \Phi(\tilde{u}, \tilde{v}) < \frac{c_2^p}{q2^p}. \]

This finishes the proof of Theorem 2.1. \( \square \)

Now, we give an application of Theorem 1.3.

**Theorem 2.2.** Assume that

(i) there exist two constants \( \overline{c} \) and \( \overline{d} \) with

\[ \max \left( 1, \frac{q^{1/p}}{2} \right) \leq \overline{c} < \overline{d} \]

such that

\[ \int_0^1 \max_{|s|+|t| \leq \overline{c}} F(x, s, t)dx < \int_\frac{1}{2}^1 F(x, \overline{d}, \overline{d})dx, \tag{2.7} \]
(ii) 
\[
\limsup_{|s|+|t| \to +\infty} \frac{F(x,s,t)}{|s|^p + |t|^q} \leq 0 \quad \text{uniformly in } X. 
\] (2.8)

Then, for each \( \lambda > \lambda_0 \), where
\[
\lambda_0 = \frac{p^p}{q^{2p}} - \frac{2^q d^q}{p} \int_0^1 \max_{|s|+|t| \leq \varepsilon} F(x,s,t) \, dx - \frac{1}{2} \int_\Omega F(x,d,d) \, dx,
\]
problem (1.1) admits at least one nontrivial weak solution \((\tilde{u}, \tilde{v})\) such that
\[
\Phi(\tilde{u}, \tilde{v}) > \frac{c^p}{q^{2p}}.
\]

**Proof.** First, note that \( \min_X \Phi = \Phi(0,0) = 0 \). Moreover, the condition (2.8) and \((F)\) implies that, for every \( \varepsilon > 0 \) there exists \( l_\varepsilon \in L^1(\Omega) \) such that
\[
F(x,s,t) \leq \varepsilon (|s|^p + |t|^q) + l_\varepsilon(x) \quad \text{for all } (x,s,t) \in \Omega \times \mathbb{R}^2.
\]
Thus
\[
\int_\Omega F(x,u,v) \, dx \leq \varepsilon (c_p ||u||_p^p + c_q ||v||_q^q) + \int_\Omega l_\varepsilon(x) \, dx \quad \text{for all } (u,v) \in X,
\]
where \( c_p \) and \( c_q \) are constants of Sobolev. Therefore
\[
I_\lambda(u,v) \geq \left( \frac{1}{p} - \varepsilon c_p \right) ||u||_p^p + \left( \frac{1}{q} - \varepsilon c_q \right) ||v||_q^q - \int_\Omega l_\varepsilon(x) \, dx.
\]
So, choosing \( \varepsilon \) small enough we deduce that \( I_\lambda \) is coercive. Our aim is to verify condition (1.8) of Theorem 1.3. Indeed, put \( u_0 \in X \) such that
\[
u_0(x) = \begin{cases} 
2d, & x \in \left[ 0, \frac{1}{2} \right], \\
1, & x \in \left[ \frac{1}{2}, 1 \right].
\end{cases}
\]
Arguing as in the proof of Theorem 2.1, putting
\[
r = \frac{c^p}{q^{2p}}.
\]
For all \((u,v) \in X\) such that \((u,v) \in \Phi^{-1}([-\infty, r])\), we have
\[
|u| + |v| \leq r,
\]
and, by (2.7) we have

$$
\rho(r) \geq \frac{\Psi(u_0, u_0) - \left(\sup_{v \in \Phi^{-1}(\{1\} \cup (r, \infty])} \Psi(v)\right)}{\Phi(u_0, u_0) - r} \\
\geq \frac{\int_1^r F(x, \overline{\alpha}, \overline{d}) \, dx - \int_1^r \max_{|s| + |t| \leq r} F(x, s, t) \, dx}{\frac{2^p}{p} \overline{d}^p - \frac{c}{q^p}} > 0.
$$

Hence, Theorem 1.3 ensures the existence of nontrivial solution \((\overline{u}, \overline{v})\) of (1.1), such that

$$
\Phi(\overline{u}, \overline{v}) > \frac{c}{q^p}
$$

This completes the proof of Theorem 2.2.

We now point out the following special cases of Theorem 2.2 when \(F\) does not depend on \(x \in [0, 1]\).

**Corollary 2.3.** Assume that \(F_u\) and \(F_v\) are nonnegative, in addition

(i) there exist two constants \(\overline{c}\) and \(\overline{d}\) with

$$
\max \left(1, \frac{q^{1/p}}{2}\right) \leq \frac{\overline{c}}{4} < \frac{\overline{d}}{4}
$$

such that

$$
\frac{F(\overline{c}, \overline{c})}{\overline{c}^p} < \frac{F(\overline{d}, \overline{d})}{2^{(2p+1)p} \overline{d}^p} \tag{2.9}
$$

(ii)

$$
\limsup_{|s| + |t| \to +\infty} \frac{F(s, t)}{|s|^p + |t|^q} = 0. \tag{2.10}
$$

Then, for each

$$
\lambda > \frac{\overline{c}^p}{q^p} - \frac{2^p}{p} \overline{d}^p
$$

problem

\[
\begin{cases}
-(|u'|^{p-2}u')' = \lambda F_u(u, v) & \text{in } (0, 1), \\
-(|v'|^{q-2}u')' = \lambda F_v(u, v) & \text{in } (0, 1), \\
u(0) = u'(1) = v(0) = v'(1) = 0,
\end{cases}
\]

admits at least one nontrivial weak solution.

**Proof.** Clearly (2.10) implies (2.8), and simple computations show that (2.9) implies (2.7). Hence Theorem 2.2 ensures the conclusion.
Example 2.4. The problem
\begin{align*}
-u'' &= \lambda \left( \frac{3}{2} v \sqrt{u} + v \sqrt{v} \right) \quad \text{in} \ (0, 1), \\
-v'' &= \lambda \left( \frac{3}{2} u \sqrt{v} + u \sqrt{u} \right) \quad \text{in} \ (0, 1), \\
\begin{align*}
u(0) &= u'(1) = v(0) = v'(1) = 0,
\end{align*}
\end{align*}

admits at least one nontrivial solution for every
\begin{equation*}
\lambda > \frac{2^{28} - 1}{2^{34} - 25}.
\end{equation*}

In fact, if we choose, for example \( c = 4, \ d = 2^{14} \) and \( F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be a function defined by
\begin{equation*}
F(s, t) = st \left( \sqrt{s} + \sqrt{t} \right),
\end{equation*}
we get
\begin{equation*}
\frac{\frac{\pi^d}{q^{2p}} - \frac{2^{28}}{p} d^d}{F(c, c) - \frac{1}{2} F(d, d)} = \frac{2^{28} - 1}{2^{34} - 25},
\end{equation*}
and all hypotheses of Corollary 2.3 are satisfied.

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References


