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Nonexistence of Limit Cycles for Planar Systems of Liénard-type

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Abstract

A new method for the nonexistence of limit cycles of the planar system including a generalized Liénard-type system is introduced. It is given by constructing a curve with some invariance defined on a half-open interval and has useful powers for the case which the equilibrium point is stable specially. Moreover, it is also applied to systems with several equilibrium points. It shall be shown that our results are used to many examples.

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1 Introduction

In this paper, a sufficient condition in order that the planar system including a generalized Liénard-type system:

$$\frac{dx}{dt} = M(x, y)[h(y) - F(x)], \qquad \frac{dy}{dt} = -N(x, y)g(x) \tag{1.1}$$

with $N(x, y) = L(x)M(x, y)$ has no limit cycles is discussed, where $F(x)$ and $g(x)$ are continuous on an open interval I which contains the origin, $M(x, y)$ and $N(x, y)$ are continuous on $I \times I$, and $h(y)$ is continuous and strictly increasing on $\mathbb{R}$. The functions $M(x, y)$, $N(x, y)$, $F(x)$, $g(x)$ and $h(y)$ satisfy smoothness conditions for the uniqueness of solutions of initial value problems.

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System (1.1) includes a classical Liénard system (the case $M(x, y) = N(x, y) = 1$ and $h(y) = y$) and plays an important role in nonlinear dynamical systems. The recent results for a Liénard polynomial system are shown in [2]. Usually, the topics of the nonexistence of limit cycles for a Liénard system have been discussed by the Bendixson–Dulac criterion in the well-known textbook such as [8]. Our aim is to give a new criterion for the nonexistence of limit cycles of system (1.1) with a stable equilibrium point specially. As the application, we also shall give a criterion for the case which the system has two equilibrium points.

Throughout this paper, we assume that

$\exists a_1 < 0$ such that such that $x(x - a_1)F(x) > 0$ for $x \neq 0$ and $x \neq a_1$,

$M(x, y) > 0$, $L(x) > 0$, $h(y)/y > 0$, $g(x)/x > 0$

for $0 < |x| < \delta$ and $\delta \geq |a_1|$.
Then we note from the linearization at the origin that it is a stable equilibrium point of system (1.1) with $\mathrm{Index}(0, 0) = +1$.

Let $h^{-1}(u)$ be the inverse function of $u = h(y)$. Note that $h^{-1}(u)$ is also strictly increasing and satisfies $uh^{-1}(u) > 0$ for $u \neq 0$, and the composite function $h^{-1}(F(x))$ is defined on an open subinterval J of I which contains the origin.

We in [4] gave the existence theorem for the homoclinic orbits of system (1.1). It has been proved by using the existence of the curves with some invariance of the system. We also want to discuss the nonexistence of limit cycles of system (1.1) by applying this idea. By constructing the curve defined on some half-open interval for system (1.1), a sufficient condition in order that the system with a stable equilibrium point has no limit cycles will be given.

Consider a function $\varphi(x)$ with the condition

$$\varphi \in C^1,\ \varphi(\alpha) = 0 \text{ and } \varphi'(x) > 0 \text{ for } x > \alpha \in (0, a_2], \tag{C1}$$

where $G(x) = \int_0^x g(\xi)d\xi$ and a_2 is a positive number satisfying the equation $G(a_2) = G(a_1)$. We remark from the condition $g(x)/x > 0$ that a_2 is unique.

Our main results are the following.

Theorem 1.1. *System* (1.1) *has no limit cycles if there exists a function* $\varphi(x)$ *with* (C1) *such that*

$$F(x) > \varphi(x) > 0 \ \text{ and } \ L(x)g(x) \leq \frac{\varphi'(x)[F(x) - \varphi(x)]}{\frac{dh}{dy}[h^{-1}(\varphi(x))]} \tag{C2}$$

for all $x \geq \alpha \in (0, a_2]$.

Corollary 1.2. *Assume the conditions in Theorem 1.1 for system* (1.1). *Then the equilibrium point* $(0, 0)$ *of the system is globally asymptotic stable.*

In the proof of the above corollary the nonexistence of limit cycles and homoclinic orbits will be used.

We shall consider the case of which system (1.1) has two equilibrium points $O(0,0)$ and $A(a^*, h^{-1}(F(a^*))$, where the function $g(x)$ satisfies the condition $g(x)/x > 0$ for $x > a^*(a^* < 0)$ or $x < a^*(a^* > 0)$. We see easily that these indices are $\mathrm{Index}\ O = +1$ and $\mathrm{Index}\ A = -1$. Thus, the nontrivial closed orbit of system (1.1) must contain the equilibrium point O only if it exists. Therefore we have from Theorem 1.1 the following.

Corollary 1.3. *Assume that system* (1.1) *has the mentioned equilibrium points O and* A *above.*

(i) *If* $a^* < a_1$*, then system* (1.1) *has no limit cycles under the conditions in Theorem 1.1.*

(ii) *If* $a_1 \le a^* \le a_2$ *and* $a^* \neq 0$*, then system* (1.1) *has no limit cycles.*

(iii) *Let* $a_2 < a^*$*. If the conditions in Theorem 1.1 are satisfied for* $\alpha \in (0, a_2]$ *and* $\alpha \le x \le a^*$*, then system* (1.1) *has no limit cycles.*

Remark 1.4. If $a^* < a_1$ and there exists a positive number β $(a_2 < \beta < \infty)$ such that the conditions in Theorem 1.1 are satisfied for $\alpha \le x \le \beta$, then we see from the above corollary that system (1.1) has a limit cycle intersecting with both the lines $x = a_1$ and $x = \beta$ if it exists.

The above results are proved in the next section. In Section 3, several examples are discussed and it will be shown from the phase portrait of the system that our methods have a power. In the final section, the criterion of the nonexistence of the limit cycle for the system with an unstable equilibrium point shall be introduced by the same discussion as Theorem 1.1.

2 Proof of Theorem 1.1

To prove Theorem 1.1, Corollary 1.2 and 1.3 we consider the following Liénard system instead of (1.1):

$$\frac{dx}{dt} = h(y) - F(x), \qquad \frac{dy}{dt} = -\tilde{g}(x), \tag{2.1}$$

where $\tilde{g}(x) = L(x)g(x)$. Note that the solution orbits of system (2.1) coincide with those ones of system (1.1). Hence we can restrict, without loss of generality, to the class of system (2.1). So we shall discuss on solution orbits in the form of system (2.1).

Aghajani and Moradifan [1] have discussed the case of $M(x,y) = a(x) = 1/N(x,y)$ in system (1.1). We note that the solution orbits of the system coincide with those ones of system (2.1) with the case $L(x) = 1/a(x)^2$. Thus, our result can apply to the example. See Section 3.

We consider a closed plane curve Γ defined by the equation $(1/2)H(y)^2 + G(x) = G(a_1)$, where $H(y) = \int_0^y h(\xi)d\xi$ and $G(x) = \int_0^x \tilde{g}(\xi)d\xi$. Let Ω be the region surrounded by Γ. Remark that $\Omega \subset \mathbb{R}^2$ is a negative invariant domain that surrounds the origin. See Figure 2.1 below.

The following facts are given by the similar discussion to the lemmas in [5].

Lemma 2.1. *A solution orbit of system* (2.1) *starting from inside* Ω *must get into* Ω.

Lemma 2.2. *System* (2.1) *has no nontrivial closed orbits inside* Ω.

From the above lemmas we have the following.

Proposition 2.3. *System* (2.1) *has a nontrivial closed orbit outside* Ω *if it exists.*

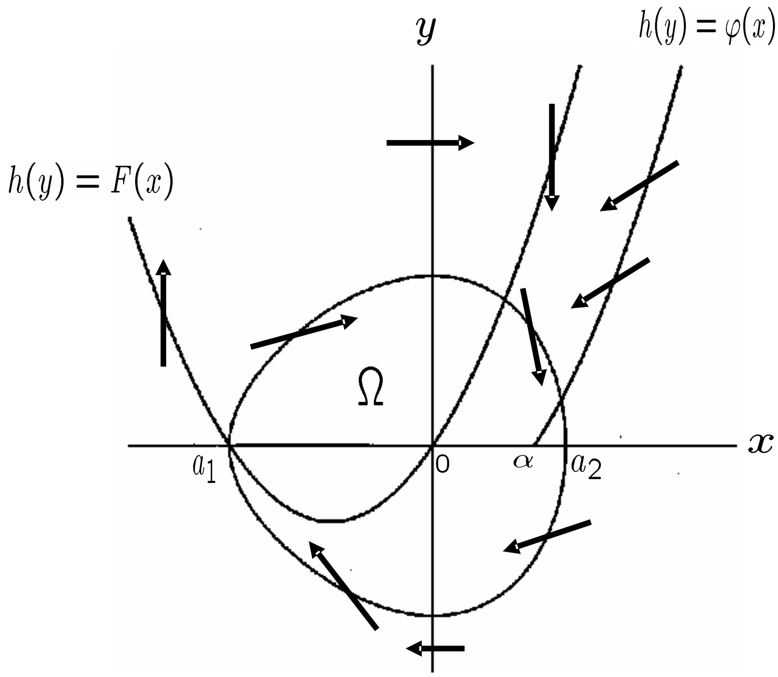

Figure 2.1: A negative invariant domain Ω for system (2.1)

Proof. Assume that system (2.1) has a nontrivial closed orbit. Then it must exist outside Ω by Proposition 2.3. Thus, if there exists a function $\varphi(x)$ with (C1) and (C2), then solution orbits of system (2.1) starting from the point $(0, q)$ $(q > 0)$ outside Ω must go into Ω or intersect the curve $h(y) = F(x)$ on the set Ω^c vertically. On the other hand, the solution orbits intersecting the curve $h(y) = F(x)$ cannot meet the curve $h(y) = \varphi(x)$ with some invariance by the condition (C2). Thus the all orbits must go into Ω. This contradicts to Proposition 2.3. Thus, the proof of Theorem 1.1 is completed now. $\square$

The proof of Corollary 1.2 is given by the same discussion as that of [6]. In facts, we can confirm the following items:

- Solution orbits of system (2.1) starting from the point $(0, q)$ $(q > 0)$ must stay into Ω.
- System (2.1) has no limit cycles.
- System (2.1) has no homoclinic orbits.

So we omit the details.

The proof of Corollary 1.3 is easily proved by the similar discussion to that of Theorem 1.1. We shall assume the existence of nontrivial closed orbit of system (2.1). Then remark that it must contain the origin, but cannot contain the equilibrium point A by the Poincaré index theorem. Thus, we see from the conditions in Theorem 1.1 that the solution orbits of system (2.1) starting from the point $(0, q)$ $(q > 0)$ outside Ω must go into Ω or the solution orbits starting from the point inside Ω must stay into Ω. This contradicts to the existence of the closed orbit from Lemma 2.2.

We note from the mentioned fact above that the conditions in Theorem 1.1 do not need for the case (ii), since the domain surrounded by a closed plane curve defined by the equation $(1/2)H(y)^2 + G(x) = G(a^*)$ is a negative invariant set including the origin.

3 Several Examples

We shall apply our results to planar systems and present the phase portrait for some system. It shall be seen that our results have a power.

Example 3.1. Consider the planar system

$$\frac{dx}{dt} = M(x, y)(y^3 - x^3 - 3x^2 - 2x), \qquad \frac{dy}{dt} = -N(x, y)x, \tag{3.1}$$

where $M(x, y) = e^x + y^2 + 1 > 0$ and $M(x, y) = 2N(x, y)$. We note that $F(x) = x(x + 1)(x + 2)$, $g(x) = x$ and $h(y) = y^3$. Taking the supplement function $\varphi(x) = (1/8)(x - 1)^3$, we see that the conditions (C1) and (C2) are satisfied for $x \geq 1$. In facts, we have $a_1 = -1$, $a_2 = \alpha = 1$,

$$F(x) - \varphi(x) = (7/8)x^3 + (27/8)x^2 + (13/8)x + (1/8) > 0$$

and

$$\begin{aligned}\frac{\varphi'(x)[F(x) - \varphi(x)]}{\dfrac{dh}{dy}[h^{-1}(\varphi(x))]} - L(x)g(x) &= \frac{1}{2}[F(x) - \varphi(x)] - \frac{1}{2}x \\ &= \frac{1}{16}(7x^3 + 27x^2 + 5x + 1) > 0\end{aligned}$$

for $x \geq 1$. Thus, we conclude from Theorem 1.1 that system (3.1) has no limit cycles. Also we see from Corollary 1.2 that the equilibrium point $(0, 0)$ is globally asymptotic stable. See Figure 3.1 below.

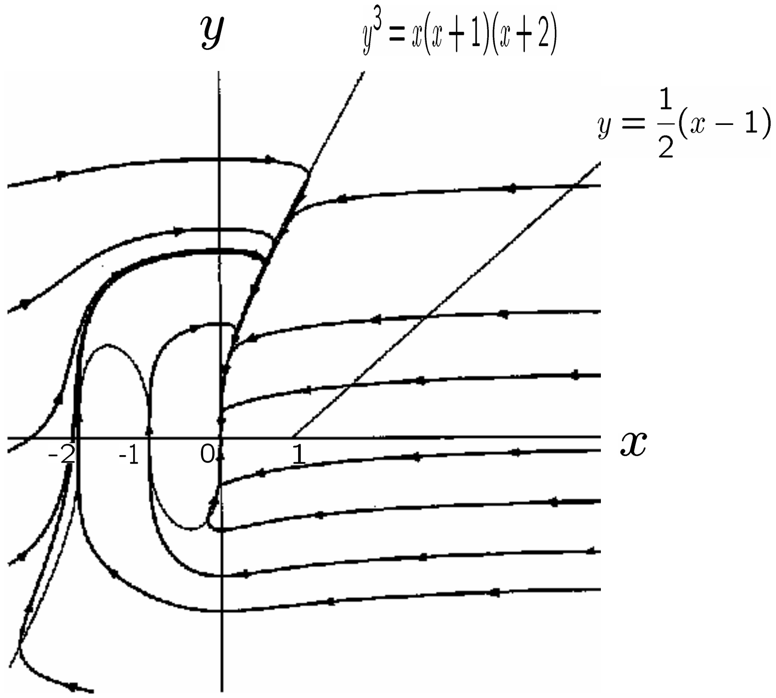

Figure 3.1: A phase portrait for system (3.1)

Example 3.2. Consider the planar system with two equilibrium points

$$\frac{dx}{dt} = -(x^2 + 2x)y^2 + y^3, \qquad \frac{dy}{dt} = -\frac{x^2 + 3x}{x^2 + 2}y^2 \tag{3.2}$$

We note that $M(x, y) = y^2 > 0$, $N(x, y) = y^2/(x^2+2)$, $F(x) = x^2+2x$, $g(x) = x^2+3x$ and $h(y) = y$. And we see that the equilibrium point $(0, 0)$ is a stable focus and $(-3, 3)$ is a saddle. Taking the supplement function $\varphi(x) = x^2 - 1$, we see that the conditions (C1) and (C2) are satisfied for $x > 0$. In facts, we have $a^* = -3 < a_1 = -2$, $a_2 > 1$, $\alpha = 1$, $N(x, y) > 0$, $F(x) - \varphi(x) = 2x + 1 > 0$ and

$$\begin{aligned} \frac{\varphi'(x)[F(x) - \varphi(x)]}{\frac{dh}{dy}[h^{-1}(\varphi(x))]} - L(x)g(x) &= \varphi'(x)[F(x) - \varphi(x)] - \frac{g(x)}{x^2 + 2} \\ &= \frac{x(4x^3 + 2x^2 + 7x + 1)}{x^2 + 2} > 0 \end{aligned}$$

for $x > 0$. Thus, we conclude from Corollary 1.3 (i) that system (3.2) has no limit cycles.

Example 3.3. Consider the following system

$$\dot{x} = \sqrt{|y|}\mathrm{sgn}(y) - F(x), \qquad \dot{y} = -g(x), \tag{3.3}$$

where $F(x) = a(|x|^n + x)$ $(a > 0)$ and $g(x) = x$. Then we have $a_1 = -1$ and $a_2 = 1$. This system is similar to that of [1]. Taking the supplement function $\varphi(x) = \alpha(|x|^{\frac{2}{3}} - \epsilon)$ $(0 < \alpha < a, 0 < \epsilon < a_2 = 1)$, we see that the system has no limit cycles if $n = 2/3$ and $a \geq (3/2)^{\frac{4}{3}}$. In facts, if $n = 2/3$, then we have $F(x) - \varphi(x) > 0$ for all $x > 0$. Let $P(\alpha) = -(4/3)\alpha^3 + (4/3)a\alpha^2 - 1$. Then, there exists $\alpha^* \in (0, (2/3)a]$ such that $P(\alpha^*) = 0$ if and only if $a \geq (3/2)^{\frac{4}{3}}$. Thus, we have $P(\alpha_1) < 0$ for $\alpha_1 \in (0, \alpha^*)$ and $P(\alpha_2) \geq 0$ for $\alpha_2 \in [\alpha^*, (2/3)a]$. We set $\alpha = \alpha_2$. For a sufficiently small ϵ, we have

$$\begin{aligned}
\frac{\varphi'(x)[F(x) - \varphi(x)]}{\dfrac{dh}{dy}[h^{-1}(\varphi(x))]} - g(x) &= 2\varphi'(x)\varphi(x)\{F(x) - \varphi(x)\} - g(x) \\
&= P(\alpha_2)x + \frac{4}{3}\alpha_2^2 a x^{\frac{4}{3}} + \frac{4}{3}\epsilon\alpha_2^2\{(2\alpha_2 - a)x^{\frac{1}{3}} - a x^{\frac{2}{3}} - \epsilon\alpha_2 x^{-\frac{1}{3}}\} > 0
\end{aligned}$$

for all $x > 0$. Thus, if $n = 2/3$ and $a \geq (3/2)^{\frac{4}{3}}$, then system (3.3) satisfies the conditions in Theorem 1.1.

Example 3.4. The following system

$$\dot{x} = m|y|^{p-1}y - F(x), \qquad \dot{y} = -g(x) \tag{3.4}$$

with $m > 0$ and $p > 1$ is the example in [7]. We shall consider the case of $F(x) = x^6 + x^3$, $g(x) = x^7$ and $p = 3$. Taking the supplement function $\varphi(x) = \alpha(x^6 - 1)$ $(0 < \alpha < 1)$, the system has no limit cycles if $m \leq 27/32$. In facts, we have $F(x) - \varphi(x) > 0$ for all $x > 0$. Let $P(\alpha) = 8\alpha(1 - \alpha)^3 - m$. Then, since there exist $\alpha^* \in (0, 1/4]$ such that $P(\alpha^*) = 0$ if and only if $m \leq 27/32$, we have $P(\alpha_1) < 0$ for $\alpha_1 \in (0, \alpha^*)$ and $P(\alpha_2) \geq 0$ for $\alpha_2 \in [\alpha^*, 1/4]$. So, letting $\alpha = \alpha_2$, we have

$$\frac{\varphi'(x)[F(x) - \varphi(x)]}{\dfrac{dh}{dy}[h^{-1}(\varphi(x))]} - g(x) > \left\{2(\frac{\alpha_2}{m})^{\frac{1}{3}}(1 - \alpha_2) - 1\right\}x^7 + 2(\frac{\alpha_2}{m})^{\frac{1}{3}}(x^4 + \alpha_2 x) > 0$$

for all $x \geq 1$. Thus, system (3.4) satisfies the conditions in Theorem 1.1.

4 The System with an unstable Equilibrium Point

In Theorem 1.1 we discussed the case of which the equilibrium point O of system (1.1) is stable. Our results can also be applied to the system with an unstable equilibrium point. We replace the condition of Section 1 to the following condition

$\exists a_2 > 0$ such that $x(x - a_2)F(x) > 0$ for $x \neq 0$ and $x \neq a_2$.

Then the origin is an unstable equilibrium point of system (1.1) with $\mathrm{Index}(0, 0) = +1$. If $F(x) \pm G(x) \to +\infty$ $(x \to \pm\infty)$, then it has been well-known from [3] that system (1.1) has at least one limit cycle. Thus, to judge the existence or nonexistence of the limit cycle for the case $F(x) \to +\infty$ $(x \to \pm\infty)$ are difficult.

By the same reason as is seen in the proof of Theorem 1.1, we can give for system (1.1) with an unstable equilibrium point the following.

Theorem 4.1. *System* (1.1) *has no limit cycles if there exists a function* $\varphi(x)$ *with* (C1) *such that*

$$F(x) > \varphi(x) > 0 \textit{ and } L(x)g(x) \geq \frac{\varphi^{'}(x)[F(x) - \varphi(x)]}{\dfrac{dh}{dy}[h^{-1}(\varphi(x))]} \tag{C3}$$

for all $x \leq \beta \in [a_1, 0)$*, where* a_1 *is a negative number satisfying the equation* $G(a_1) = G(a_2)$.

We shall prove the proof of Theorem 4.1 by the same discussion as Theorem 1.1. First, we easily see that Proposition 2.3 holds for system (2.1), where Ω is a positive invariant domain surrounded by the closed curve Γ. Assume that system (2.1) has a nontrivial closed orbit. Then it must contain the origin and exist outside Ω by Proposition 2.3. Thus, if there exists a function $\varphi(x)$ with the condition

$\varphi \in C^1$, $\varphi(\beta) = 0$ and $\varphi^{'}(x) > 0$ for $x < \beta \in [a_1, 0)$

and (C3), then the solution orbits of system (2.1) starting from the point $(0, q)$ $(q > 0)$ inside Ω must go out Ω and intersect the curve $h(y) = F(x)$ on the set Ω^c vertically. However, they cannot intersect the curve $h(y) = \varphi(x)$ defined by the condition (C3). This contradicts to the existence of the nontrivial closed orbit. Thus, the proof of Theorem 4.1 is completed now.

For system (1.1) with two equilibrium points O and A in Section 1 we have the following.

Corollary 4.2. *Assume that system* (1.1) *has two equilibrium points O and* A.

(iv) *If* $a_2 < a^*$*, then system* (1.1) *has no limit cycles under the conditions in Theorem 4.1.*

(v) *If* $a_1 \leq a^* \leq a_2$ *and* $a^* \neq 0$*, then system* (1.1) *has no limit cycles.*

(vi) *Let* $a^* < a_1$*. If the conditions in Theorem 4.1 are satisfied for* $\beta \in [a_1, 0)$ *and* $a^* \leq x \leq \beta$*, then system* (1.1) *has no limit cycles.*

Outline of the proof of Corollary 4.2 is same as that of Corollary 1.3. So we omit the details.

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