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A solid Foundation for q-Appell Polynomials

Thomas Ernst
Uppsala University
Department of Mathematics
P.O. Box 480, SE-751 06 Uppsala, Sweden
thomas@math.uu.se

Abstract

The term multiplicative q-Appell polynomial is introduced, with immediate interesting generalizations for q-analogues of Nørlunds formulas. We show that the set of q-Appell polynomials is a commutative ring. Finally, we show some formulas for q-Geronimus Appell polynomials, generalizing a paper of Geronimus from 1934.

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1 Introduction

The purpose of this article is to generalize the previous expansion formulas from [5] to multiplicative q-Appell polynomials. The multiplicative q-Appell polynomials possess a certain multiplicative condition on the generating function. We will also put the calculations for q-Appell polynomials on a solid algebraic basis, establishing a link between algebra and special functions. Many calculations on Appell polynomials today are made without knowing that they are, in fact, Appell polynomials. This means that our article will also be of great use for scientists working with the undeformed case $q = 1$.

This paper is organized as follows: In this section we give a short introduction and make some definitions. In section two we show that the q-Appell polynomials (and the q-Appell numbers as well) form a commutative ring with two operations. $\bigoplus$ and $\bigodot$, induced from + and $\oplus_q$. In section three we discuss multiplicative q-Appell polynomials. We will define three kinds of multiplicative q-Appell polynomials, for the purpose of defining NWA and JHC triangle operators, and deriving complementary argument theorems and Nørlund [7] type expansion formulas. Some of the proofs are the same

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as in [1]. In an appendix we introduce the very general q-Geronimus Appell polynomials, which contain an extra variable y, and show some remarkable symmetry properties. We now start with the definitions, compare with the book [2]. Some of the notation is well-known and will be skipped.

Definition 1.1. Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\{n\}_q!}{\{k\}_q!\{n-k\}_q!}, k = 0, 1, \ldots, n. \tag{1.1}$$

Let a and b be any elements with commutative multiplication. Then the NWA q-addition is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n = 0, 1, 2, \ldots. \tag{1.2}$$

If $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function is defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{1.3}$$

The q-derivative is defined by

$$(\mathrm{D}_q f)(x) \equiv \begin{cases} \frac{f(x)-f(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \frac{df}{dx}(x) & \text{if } q = 1; \\ \frac{df}{dx}(0) & \text{if } x = 0. \end{cases} \tag{1.4}$$

Theorem 1.2. *We have the q-Taylor formula*

$$\Phi^{(n)}_{\nu,q}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(n)}_{\nu-k,q}(x) y^k. \tag{1.5}$$

Definition 1.3. For every power series $f_n(t)$, the q-Appell polynomials or Φ_q polynomials of degree ν and order n have the following generating function:

$$f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu,q}(x). \tag{1.6}$$

For $x = 0$ we get the $\Phi^{(n)}_{\nu,q}$ number of degree ν and order n.

Definition 1.4. Under the assumption that the function $h(t)^n$ can be expressed analytically in $\mathbb{R}[[t]]$, and for $f_n(t)$ of the form $h(t)^n$, we call the q-Appell polynomial Φ_q in (1.6) *multiplicative*.

$$\mathrm{E}_q(xt)h(t)^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\mathcal{M},\nu,q}(x),\ n \in \mathbb{N}. \tag{1.7}$$

Then we have

Theorem 1.5. *If* $\sum_{l=1}^{s} n_l = n,\ n \in \mathbb{N}$,

$$\Phi^{(n)}_{\mathcal{M},k,q}(x_1 \oplus_q \ldots \oplus_q x_s) = \sum_{m_1+\ldots+m_s=k} \binom{k}{m_1,\ldots,m_s}_q \prod_{j=1}^{s} \Phi^{(n_j)}_{\mathcal{M},m_j,q}(x_j), \tag{1.8}$$

where we assume that n_j *operates on* x_j.

Two special cases of multiplicative q-Appell polynomials are NWA multiplicative q-Appell polynomials $\Phi^{(n)}_{\text{NWA},\mathcal{M},\nu,q}(x)$ and JHC multiplicative q-Appell polynomials $\Phi^{(n)}_{\text{JHC},\mathcal{M},\nu,q}(x)$, which are defined as follows:

Definition 1.6.

$$\mathrm{E}_q(xt)h_1(t)^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\text{NWA},\mathcal{M},\nu,q}(x), n \in \mathbb{N}, \tag{1.9}$$

where the only q-exponential function in $h_1(t)$ is $\mathrm{E}_q(t)$.

Definition 1.7.

$$\mathrm{E}_q(xt)h_2(t)^n = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\text{JHC},\mathcal{M},\nu,q}(x), n \in \mathbb{N}, \tag{1.10}$$

where the only q-exponential function in $h_2(t)$ is $\mathrm{E}_{\frac{1}{q}}(t)$.

We assume that $\Phi^{(n)}_{\text{NWA},\mathcal{M},\nu,q}(x)$ and $\Phi^{(n)}_{\text{JHC},\mathcal{M},\nu,q}(x)$ always occur together. The function $h_2(t)$ is equal to an inversion of the basis q in $h_1(t)$.

Definition 1.8. Assume that $n \in \mathbb{Z}$.
We define the operator $\triangle_{\mathcal{M},q}$ by

$$\triangle_{\mathcal{M},q}\Phi^{(n)}_{\mathcal{M},\nu,q}(x) = \mathrm{D}_q\Phi^{(n-1)}_{\mathcal{M},\nu,q}(x). \tag{1.11}$$

The operators $\triangle_{\mathcal{M},q}$ and D_q commute since D_q maps a q-Appell polynomial to another q-Appell polynomial.

2 The Ring of q-Appell Polynomials

Definition 2.1. We denote the set of all q-Appell polynomials (in the variable x) by $\mathcal{A}_{x;q}$.

Let $\Phi_{n,q}(x)$ and $\Psi_{n,q}(x)$ be two elements in $\mathcal{A}_{x;q}$. Then the operations $\bigoplus$ and $\bigodot$ are defined as follows:

$$(\Phi_q(x) \bigoplus \Psi_q(x))_n \equiv (\Phi_q(x) + \Psi_q(x))_n, \tag{2.1}$$

$$(\Phi_q(x) \bigodot \Psi_q(x))_n \equiv (\Phi_q(x) \oplus_q \Psi_q(x))^n = \sum_{k=0}^{n} \binom{n}{k}_q \Phi_{n-k,q}(x)\Psi_{k,q}(x). \tag{2.2}$$

We keep the usual priority between $\bigoplus$ and $\bigodot$.

Theorem 2.2. *$\mathcal{A}_{x;q}(\bigoplus, \bigodot)$ is a commutative ring.*

Proof. We presume that $\Phi_{n,q}(x)$, $\Xi_{n,q}(x)$ and $\Psi_{n,q}$(x) are three elements in $\mathcal{A}_q$ corresponding to the generating functions $f(t)$, $g(t)$ and $h(t)$ respectively.

We first show that $\bigoplus$ is well-defined. Assume that $f(0) + h(0) \neq 0$. Then

$$\Phi_q(x) \bigoplus \Psi_q(x) \in \mathcal{A}_{x;q}, \tag{2.3}$$

$$\Phi_{n,q}(x) \bigoplus \Psi_{n,q}(x) \text{ has generating function } f(t) + h(t). \tag{2.4}$$

The associative law for $\bigoplus$ reads:

$$\left(\Phi_q(x) \bigoplus (\Xi_q(x) \bigoplus \Psi_q(x))\right)_n = \left((\Phi_q(x) \bigoplus \Xi_q(x)) \bigoplus \Psi_q(x)\right)_n. \tag{2.5}$$

This follows from the associativity of $+$.

The commutative law for $\bigoplus$ reads:

$$\left(\Phi_q(x) \bigoplus \Psi_q(x)\right)_n = \left(\Psi_q(x) \bigoplus \Phi_q(x)\right)_n. \tag{2.6}$$

This follows from the commutativity of $+$.

The identity element with respect to $\bigoplus$ is the sequence $\mathcal{O}$. We have

$$\left(\Phi_q(x) \bigoplus \mathcal{O}\right)_n = (\Phi_q(x))_n. \tag{2.7}$$

There exists $-\Phi_q(x)$ such that

$$\left(\Phi_q(x) \bigoplus (-\Phi_q(x))\right)_n = (\mathcal{O})_n. \tag{2.8}$$

This follows from the corresponding property of real numbers.

Then we show that $\bigodot$ is well-defined. We have

$$\Phi_q(x) \bigodot \Psi_q(x) \in \mathcal{A}_{x;q}, \tag{2.9}$$

$$\Phi_{n,q}(x) \bigodot \Psi_{n,q}(x) \text{ has generating function } f(t)h(t). \tag{2.10}$$

This follows since in the umbral sense

$$f(t)h(t) = \mathrm{E}_q(t(\Phi_q \oplus_q \Psi_q)). \tag{2.11}$$

The identity element with respect to $\bigodot$ is $\mathrm{I} \equiv \delta_{n,0}$. This corresponds to $f(t) = 1$. To show this, we put $k = n$ and $\Phi_q = 1$ in (2.2). We have

$$\left(\Phi_q(x) \bigodot \mathrm{I}\right)_n = \left(\mathrm{I} \bigodot \Phi_q(x)\right)_n = \Phi_{n,q}(x). \tag{2.12}$$

The associative law for $\bigodot$ reads:

$$\left(\Phi_q(x) \bigodot (\Xi_q(x) \bigodot \Psi_q(x))\right)_n = \left((\Phi_q(x) \bigodot \Xi_q(x)) \bigodot \Psi_q(x)\right)_n. \tag{2.13}$$

This follows from the associativity of $\oplus_q$.

The commutative law for $\bigodot$ reads:

$$\left(\Phi_q(x) \bigodot \Psi_q(x)\right)_n = \left(\Psi_q(x) \bigodot \Phi_q(x)\right)_n. \tag{2.14}$$

This follows from the commutativity of $\oplus_q$.

The distributive law reads:

$$\begin{aligned} &\left(\Phi_q(x) \bigodot (\Xi_q(x) \bigoplus \Psi_q(x))\right)_n \\ &= \left(\Phi_q(x) \bigodot \Xi_q(x) \bigoplus \Phi_{n,q}(x) \bigodot \Psi_q(x)\right)_n. \end{aligned} \tag{2.15}$$

This follows from the following computation:

$$\begin{aligned} &\left(\Phi_q(x) \bigodot (\Xi_q(x) \bigoplus \Psi_q(x))\right)_n \\ &\overset{\text{by}((2.2))}{=} \sum_{k=0}^{n} \binom{n}{k}_q \Phi_{n-k,q}(x) \left(\Xi_q(x) \bigoplus \Psi_q(x)\right)_k \\ &\overset{\text{by}((2.1))}{=} \sum_{k=0}^{n} \binom{n}{k}_q \Phi_{n-k,q}(x) \left(\Xi_q(x) + \Psi_q(x)\right)_k. \end{aligned} \tag{2.16}$$

On the other hand,

$$\begin{aligned}
&\Big(\Phi_q(x)\bigodot\Xi_q(x)\bigoplus\Phi_q(x)\bigodot\Psi_q(x)\Big)_n\\
&\overset{\text{by}((2.1))}{=}\Big(\Phi_q(x)\bigodot\Xi_q(x)+\Phi_q(x)\bigodot\Psi_q(x)\Big)_n\\
&\overset{\text{by}((2.2))}{=}\sum_{k=0}^{n}\binom{n}{k}_q\Phi_{n-k,q}(x)\left(\Xi_q(x)+\Psi_q(x)\right)_k.
\end{aligned}\tag{2.17}$$

Both expressions are equal. □

Corollary 2.3. *The set of all q-Appell numbers $\mathcal{A}_q$ is a commutative ring. It is also a subring of $\mathcal{A}_{x;q}$.*

Proof. Put $x=0$ in the previous theorem. □

3 Multiplicative q-Appell Polynomials

Assume that $n,\ p\in\mathbb{Z}$. Then all the following formulas from [2] also hold for general multiplicative q-Appell polynomials.

Theorem 3.1.

$$\Phi_{\mathcal{M},\nu,q}^{(-n-p)}(x\oplus_q y)\doteq(\Phi_{\mathcal{M},q}^{(-n)}(x)\oplus_q\Phi_{\mathcal{M},q}^{(-p)}(y))^{\nu}.\tag{3.1}$$

A special case is the following formula:

$$\Phi_{\mathcal{M},\nu,q}^{(-n)}(x\oplus_q y)\doteq(\Phi_{\mathcal{M},q}^{(-n)}(x)\oplus_q y)^{\nu}.\tag{3.2}$$

Theorem 3.2. *A recurrence formula for the NWA multiplicative q-Appell numbers: If $n,p\in\mathbb{Z}$ then*

$$\Phi_{\mathcal{M},\nu,q}^{(n+p)}\doteq(\Phi_{\mathcal{M},q}^{(n)}\oplus_q\Phi_{\mathcal{M},q}^{(p)})^{\nu},\tag{3.3}$$

Theorem 3.3.

$$(x\oplus_q y)^{\nu}\doteq(\Phi_{\mathcal{M},q}^{(-n)}(x)\oplus_q\Phi_{\mathcal{M},q}^{(n)}(y))^{\nu},\tag{3.4}$$

Proof. Put $p=-n$ in (3.1). □

In particular for $y=0$, we obtain

$$x^{\nu}\doteq(\Phi_{\mathcal{M},q}^{(-n)}\oplus_q\Phi_{\mathcal{M},q}^{(n)}(x))^{\nu}.\tag{3.5}$$

These recurrence formulas express NWA multiplicative q-Appell polynomials of order n without mentioning polynomials of negative order.

This can be expressed in another form:

$$x^{\nu}=\sum_{s=0}^{\nu}\frac{\Phi_{\mathcal{M},s,q}^{(-n)}}{\{s\}_q!}\mathrm{D}_q^s\Phi_{\mathcal{M},\nu,q}^{(n)}(x), \tag{3.6}$$

We conclude that NWA multiplicative q-Appell polynomials satisfy linear q-difference equations with constant coefficients.

The following formula is useful for the computation of multiplicative q-Appell polynomials of positive order. This is because the polynomials of negative order are of simpler nature and can easily be computed. When the $\Phi_{\mathcal{M},s,q}^{(-n)}$ etc. are known, (3.7) can be used to compute the $\Phi_{\mathcal{M},s,q}^{(n)}$.

Theorem 3.4.

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_q\Phi_{\mathcal{M},s,q}^{(n)}\Phi_{\mathcal{M},\nu-s,q}^{(-n)}=\delta_{\nu,0}. \tag{3.7}$$

Proof. Put $x=y=0$ in (3.4). □

Theorem 3.5. *A generalization of [2, 4.261]: Under the assumption that $f(x)$ is analytic with q-Taylor expansion*

$$f(x)=\sum_{\nu=0}^{\infty}\mathrm{D}_q^{\nu}f(0)\frac{x^{\nu}}{\{\nu\}_q!}, \tag{3.8}$$

we can express powers of $\triangle_{\mathcal{M},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough:

$$\triangle_{\mathcal{M},q}^{n}f(x)=\sum_{\nu=0}^{\infty}\mathrm{D}_q^{\nu+n}f(0)\frac{\Phi_{\mathcal{M},\nu,q}^{(-n)}(x)}{\{\nu\}_q!},\ n>0. \tag{3.9}$$

Proof. Simply use formula (1.11). □

4 Appendix: The Geronimus Appell Polynomials

Yakov Lazarevich Geronimus (1898–1984) obtained his thesis from Charkov State University in 1939 on the subject approximations and expansions. His tutor was the famous Sergei Natanovich Bernstein (1880–1968), well known for the Bernstein polynomials. To explain the connection with Appell polynomials we have to go back to Bernstein's student years in Paris. Bernstein submitted his thesis at Sorbonne in the spring of 1904, supervised by Hilbert and Picard. Probably Bernstein learned about Appell polynomials during this time, and he passed on this knowledge to Geronimus. He could also have been influenced by his countryman Vashchenko-Zakharchenko (1825–1912), who had studied umbral calculus in Paris 1847-48 [2, p. 69]. We are now going to find the q-analogues of the first formulas in Geronimus article [6]. We first define a very general q-difference operator.

Definition 4.1. Let $h(x.n)$ be a function of two variables. The operator $\triangle_{x,n;q}$ is defined by

$$\triangle_{x,h;q}\left(q^{h(x.n)}\binom{x}{n}_q\right)=q^{h(x.n-1)}\binom{x}{n-1}_q. \tag{4.1}$$

Definition 4.2. A q-analogue of [6, p. 13, (1)]. The generating function for the q-Geronimus Appell polynomials $\Omega_{\nu,q}(x,y,f)$ is given by

$$f(t)E_q(xt)\sum_{l=0}^{\infty}t^l\binom{y}{l}_q q^{h(y.l)}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\Omega_{\nu,q}(x,y,f), \tag{4.2}$$

$f(t)$ a given function

$$f(t)=\sum_{k=0}^{\infty}\frac{f_kt^k}{\{k\}_q!},\ f_k\text{ constants}. \tag{4.3}$$

We obtain the fundamental property

Theorem 4.3. *A q-analogue of [6, p. 13, (4)]*

$$\triangle_{y,h;q}\Omega_{\nu,q}(x,y,f)=\{\nu\}_q\Omega_{\nu-1,q}(x,y,f). \tag{4.4}$$

Proof. Operate with $\triangle_{y,h;q}$ on (4.2). □

We observe that $\Omega_{\nu,q}(x,y,f)$ contains an extra variable y. The relation (4.4) is very similar to the defining relation for q-Appell polynomials.

Corollary 4.4. *A q-analogue of [6, p. 13, (6)]*

$$\frac{\Omega_{n,q}(x,y,f)}{\{n\}_q!}=\sum_{k=0}^{n}\binom{y}{k}_q q^{h(y,k)}\frac{(x\oplus_q f)^{n-k}}{\{n-k\}_q!}. \tag{4.5}$$

5 Discussion

We have generalized some of the previous formulas from [1] and [2] to multiplicative q-Appell polynomials. This calculus should be adapted to special cases of Appell polynomials, which will hopefully appear in the future. The fact that q-Appell polynomials is a ring puts the subject on a solid mathematical basis and improves the characterization in [3]. In fact this is a special case of the matrix q-exponential function (for the q-Polya-Vein matrix [4, p. 1178]).

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