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Limit-point/limit-circle Results for Forced Second Order Differential Equations

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Abstract

In this paper, the authors consider both the forced and unforced versions of a second order nonlinear differential equation with a p-Laplacian. They give sufficient conditions as well as necessary and sufficient conditions for the unforced equation to be of the strong nonlinear limit-circle type, and they give sufficient conditions for the forced equation to be of the strong nonlinear limit-circle type. They also give sufficient conditions for the forced equation to not be of the strong nonlinear limit-circle type.

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1 Introduction

In this paper, we study the forced second order nonlinear differential equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda}\operatorname{sgn} y = e(t) \tag{1.1}$$

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as well as the unforced equation

$$\big(a(t)|y'|^{p-1}y'\big)' + r(t)|y|^{\lambda}\operatorname{sgn} y = 0, \tag{1.2}$$

where $\mathbb{R}_+ = [0,\infty)$, $\lambda > p > 0$, $a \in C^1(\mathbb{R}_+)$, $r \in C^1(\mathbb{R}_+)$, $e \in C(\mathbb{R}_+)$, and $a(t) > 0$ and $r(t) > 0$ on $\mathbb{R}_+$. If $\lambda = p$, then these are the well-known *half-linear* equations. Since here we have $\lambda > p$, we are in what is referred to as the *super-half-linear* case; if $\lambda < p$, then it is the *sub-half-linear* case. This terminology was introduced for the first time in [2–4].

Remark 1.1. The functions a, r, and e are smooth enough so that all solutions of (1.1) and (1.2) are defined on $\mathbb{R}_+$ (e.g., see [6, Lemma 1(i)]). Moreover, all nontrivial solutions of (1.1) are nontrivial in any neighborhood of ∞ (e.g., see [14, Theorem 9.4] and [8, Theorem 4]).

It will be convenient to define the following constants:

$$\begin{gathered}
\alpha = \frac{p+1}{(\lambda+2)p+1}\,, \quad \beta = \frac{(\lambda+1)p}{(\lambda+2)p+1}\,, \quad \gamma = \frac{p+1}{p(\lambda+1)}\,, \\
\delta = \frac{p+1}{p}\,, \quad \delta_1 = \gamma^{-\frac{1}{\lambda+1}}\,, \\
\omega = \frac{p}{(\lambda+2)p+1}\,, \quad \omega_1 = \frac{(\lambda+2)p+1}{(\lambda+1)(p+1)} < 1\,, \\
\omega_2 = \frac{(\lambda+1)(p+1)}{\lambda-p}\,, \quad \omega_3 = \frac{(\lambda+p+2)p}{(p+1)((\lambda+2)p+1)}\,, \\
q = \max\left\{\frac{1}{p+1}, \omega_1\right\} < 1\,, \quad q_1 = \frac{1}{1-q}\,.
\end{gathered}$$

Notice that $\alpha = 1-\beta$. We define the functions $R, g\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$R(t) = a^{1/p}(t)\,r(t) \quad \text{and} \quad g(t) = -\frac{a^{1/p}(t)R'(t)}{R^{\alpha+1}(t)}\,.$$

For any continuous function $h\colon \mathbb{R}_+ \to \mathbb{R}$, we let $h_+(t) = \max\{h(t),0\}$ and $h_-(t) = \max\{-h(t),0\}$ so that $h(t) = h_+(t) - h_-(t)$. In order to simplify some of the notation in what follows, for any solution $y\colon \mathbb{R}_+ \to \mathbb{R}$ of (1.1), we let

$$\begin{gathered}
y^{[1]}(t) = a(t)|y'(t)|^{p-1}y'(t)\,, \\
V(t) = \frac{|y^{[1]}(t)|^{\delta}}{R(t)} + \gamma|y(t)|^{\lambda+1} = \frac{a(t)}{r(t)}|y'(t)|^{p+1} + \gamma|y(t)|^{\lambda+1}\,, \\
F(t) = R^{\beta}(t)\,V(t)\,, \qquad (1.3)
\end{gathered}$$

$$A_1(t) = \big(\max_{0\le s\le t}|g(s)| + 1\big)\int_0^t R^{-\omega}(s)|e(s)|\,ds\,, \tag{1.4}$$

$$A(t) = \int_0^t |g'(s)|\, ds + 1 + \big(\max_{0\le s\le t} |g(s)| + 1\big) \int_0^t R^{-\omega}(s)|e(s)|\, ds\,, \tag{1.5}$$

and

$$G(t) = F(t)\, A^{-q_1}(t)\,. \tag{1.6}$$

Here we are especially interested in studying what are known as the strong nonlinear limit-circle type solutions of equations (1.1) and (1.2) as defined below. This property was first introduced in [4] for sub-half-linear equations and in [3] for super-half-linear equations.

Definition 1.2. A solution y of (1.1) is said to be of the strong nonlinear limit-circle type if

$$\int_0^\infty |y(t)|^{\lambda+1} dt < \infty \quad \text{and} \quad \int_0^\infty \frac{|y^{[1]}(t)|^\delta}{R(t)}\, dt = \int_0^\infty \frac{a(t)}{r(t)} |y'(t)|^{p+1} dt < \infty\,.$$

Equation (1.1) is said to be of the strong nonlinear limit-circle type if every solution is of the strong nonlinear limit-circle type.

2 Preliminary Lemmas

In this section, we present some lemmas that will be needed in the proofs of our main results. Our first lemma provides some basic information about the behavior of solutions of equation (1.1).

Lemma 2.1. *Let y be a solution of* (1.1). *Then:*

(i) *the estimates*

$$|y(t)| \le \delta_1 R^{-\omega}(t) F^{\frac{1}{\lambda+1}}(t) \tag{2.1}$$

and

$$|y^{[1]}(t)| \le R^{\omega}(t) F^{\frac{p}{p+1}}(t) \tag{2.2}$$

hold for $t \ge 0$;

(ii) *for $0 \le \tau < t$, we have*

$$\begin{aligned} F(t) = F(\tau) &- \alpha g(\tau)\, y(\tau) y^{[1]}(\tau) + \alpha g(t)\, y(t) y^{[1]}(t) \\ &- \alpha \int_\tau^t g'(s)\, y(s) y^{[1]}(s)\, ds + I(t,\tau), \end{aligned} \tag{2.3}$$

where

$$I(t,\tau)=\int_\tau^t \big[\delta R^{-\alpha}(s)|y^{[1]}(s)|^{1/p}\operatorname{sgn} y^{[1]}(s)-\alpha g(s)\,y(s)\big]\,e(s)\,ds\,.$$

Moreover,

$$|I(t,\tau)|\le \int_\tau^t R^{-\omega}(s)\big[\delta F^{1/(p+1)}(s)+\alpha\delta_1|g(s)|F^{1/(\lambda+1)}(s)\big]\,|e(s)|\,ds\,. \tag{2.4}$$

Proof. The estimates (2.1), (2.2), and (2.3) were proved in [6, Lemma 1]; (2.4) follows from these. □

Next, we need to show that the function G is bounded.

Lemma 2.2. *For any solution y of* (1.1)*, the function G is bounded.*

Proof. Let y be a solution of (1.1). Since $G(t)\ge 0$ on $\mathbb{R}_+$, we only need to show that G is bounded from above. So suppose that this is not the case; then there is a sequence $\{t_k\}_{k=1}^\infty$ such that $\lim\limits_{k\to\infty} t_k=\infty$ and

$$\lim_{k\to\infty} G(t_k)=\infty\,. \tag{2.5}$$

It follows from (1.6) that $\lim\limits_{k\to\infty} F(t_k)=\infty$, so we may assume that $F(t_k)\ge 1$ for $k\ge 1$. Hence, there exist two sequences $\{\sigma_k\}_{k=1}^\infty$ and $\{\tau_k\}_{k=1}^\infty$ with $t_k\le\sigma_k\le\tau_k$ such that

$$G(t_k)=G(\sigma_k)=\frac{1}{2}G(\tau_k) \tag{2.6}$$

and

$$G(\sigma_k)\le G(t)\le G(\tau_k)\quad\text{for}\quad \sigma_k\le t\le\tau_k\,,\ \ k=1,2,\dots \tag{2.7}$$

From (1.6) and (2.7), we see that

$$\max_{\sigma_k\le t\le\tau_k} F(t)=F(\tau_k)\quad\text{and}\quad F(t)\ge 1\quad\text{on}\quad [\sigma_k,\tau_k]\,. \tag{2.8}$$

Now, g is locally of bounded variation, so

$$|g(t)|-|g(0)|\le|g(t)-g(0)|\le\int_0^{\tau_k}|g'(s)|\,ds \tag{2.9}$$

for $t\in[\sigma_k,\tau_k]$. Furthermore, Lemma 2.1(i) and (2.8) imply the existence of $k_0\in\{1,2,\dots\}$ and $C>0$ (not depending on k) such that

$$\begin{aligned}
|I(\tau_k,\sigma_k)| &\le \int_{\sigma_k}^{\tau_k} R^{-\omega}(s)\big(\delta F^{1/(p+1)}(s)+\alpha\delta_1|g(s)|F^{1/(\lambda+1)}(s)\big)\,|e(s)|\,ds\\
&\le \big[\delta+\alpha\delta_1\max_{0\le s\le\tau_k}|g(s)|\big]\int_{\sigma_k}^{\tau_k} R^{-\omega}(s)F^{1/(p+1)}(s)|e(s)|\,ds\\
&\le C\big(\max_{0\le s\le\tau_k}|g(s)|+1\big)F^{1/(p+1)}(\tau_k)\int_{\sigma_k}^{\tau_k} R^{-\omega}(s)|e(s)|\,ds\\
&\le CA_1(\tau_k)F^{1/(p+1)}(\tau_k)
\end{aligned} \tag{2.10}$$

for $k \ge k_0$.

Letting $\tau = \sigma_k$ and $t = \tau_k$ in part (ii) of Lemma 2.1 and using (2.1), (2.2), (2.8), (2.9), and (2.10), we obtain

$$|y(t)y^{[1]}(t)| \le \delta_1 F^{\omega_1}(\tau_k) \quad \text{for} \quad t \in [\sigma_k, \tau_k]$$

and

$$\begin{aligned} F(\tau_k) - F(\sigma_k) &\le \alpha_1 F^{\omega_1}(\tau_k)\Big(|g(\sigma_k)| + |g(\tau_k)| + \int_0^{\tau_k} |g'(s)|\,ds\Big) \\ &\quad + C A_1(\tau_k) F^{1/(p+1)}(\tau_k) \\ &\le F^q(\tau_k)\Big\{2\alpha_1|g(0)| + 3\alpha_1 \int_0^{\tau_k} |g'(s)|\,ds + C A_1(\tau_k)\Big\} \end{aligned} \tag{2.11}$$

for $k \ge k_0$, where $\alpha_1 = \alpha\gamma^{-1/(\lambda+1)}$. It is easy to see that there is an integer $k_1 \ge k_0$ and a constant $M > 0$ such that

$$2\alpha_1|g(0)| \le M\Big(\int_0^{\tau_k} |g'(s)|\,ds + 1\Big) \quad \text{for} \quad k \ge k_1 .$$

From, (1.4), (1.5), and (2.11), we obtain

$$F(\tau_k) - F(\sigma_k) \le M_1 A(\tau_k) F^q(\tau_k)$$

for $k \ge k_1$, where $M_1 = \max\{C, M + 3\alpha_1\}$. From this, (1.6), and (2.6), we have

$$\begin{aligned} \frac{1}{2}A^{q_1}(\tau_k)G(\tau_k) &= A^{q_1}(\tau_k)\big[G(\tau_k) - G(\sigma_k)\big] \le A^{q_1}(\tau_k)G(\tau_k) - A^{q_1}(\sigma_k)G(\sigma_k) \\ &= F(\tau_k) - F(\sigma_k) \le M_1 A^{1+qq_1}(\tau_k) G^q(\tau_k), \end{aligned}$$

or

$$G^{1-q}(\tau_k) \le 2M_1$$

for $k \ge k_1$. This contradicts (2.5) and completes the proof of the lemma. □

The following result is well known in the case $m = 2$, however we will need it for other not necessarily integer values.

Lemma 2.3. *Let $m > 0$, $f \in C^1(\mathbb{R}_+)$, f' be bounded on $\mathbb{R}_+$, and*

$$\int_0^\infty |f(t)|^m\,dt < \infty . \tag{2.12}$$

Then

$$\lim_{t\to\infty} f(t) = 0 . \tag{2.13}$$

Proof. Let $M > 0$ be such that

$$|f'(t)| \le M < \infty \quad \text{on} \quad \mathbb{R}_+\,, \tag{2.14}$$

$\limsup_{t\to\infty} f(t) = N$, and $\liminf_{t\to\infty} f(t) = N_1$, for some N and N_1 with $-\infty \le N_1 \le N \le \infty$. Suppose that (2.13) does not hold. Then, in view of (2.12), we see that $-\infty \le N_1 \le 0$, $0 \le N \le \infty$, and we do not have $N = N_1 = 0$.

If $N > 0$, then there are sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\bar t_n\}_{n=1}^{\infty}$ such that

$$t_n < \bar t_n < t_{n+1}\,, \quad f(t_n) = \frac{N}{2}\,, \quad \text{and} \quad \bar t_n = t_n + \frac{N}{2M} \quad \text{for} \quad n = 1, 2, \ldots .$$

Now (2.14) implies $-M \le f'(t)$, so the function f lies above the line passing through the points $\big[t_n, \frac{N}{2}\big]$ and $[\bar t_n, 0]$, i.e.,

$$f(t) \ge \frac{N}{2} - M(t - t_n) \ge 0 \quad \text{for} \quad t \in [t_n, \bar t_n]\,.$$

Hence,

$$\int_0^\infty |f(t)|^m\,dt \ge \sum_{n=1}^\infty \int_{t_n}^{\bar t_n} [f(t)]^m\,dt \ge \sum_{n=1}^\infty \int_{t_n}^{\bar t_n} \Big[\frac{N}{2} - M(t-t_n)\Big]^m dt$$
$$= \sum_{n=1}^\infty \frac{1}{M(m+1)}\Big(\frac{N}{2}\Big)^{m+1} = \infty\,,$$

contradicting (2.12).

If $N = 0$, then $N_1 < 0$, and a similar argument will again yield a contradiction. □

3 Strong Nonlinear Limit-circle Results

Our first result in this section gives sufficient conditions for equation (1.2) to be of the strong nonlinear limit-circle type, as well as a necessary and sufficient condition for this to happen.

Theorem 3.1. (i) *If*

$$\int_0^\infty R^{-\beta}(t)\Big(\int_0^t |g'(s)|\,ds + 1\Big)^{\omega_2} dt < \infty, \tag{3.1}$$

then equation (1.2) *is of the strong nonlinear limit-circle type.*

(ii) *Assume that*

$$\int_0^\infty |g'(s)|\,ds < \infty. \tag{3.2}$$

Then equation (1.2) *is of the strong nonlinear limit-circle type if and only if*

$$\int_0^\infty R^{-\beta}(t)\,dt < \infty\,. \tag{3.3}$$

Proof. (i) Let y be a solution of (1.2). If g is not identically constant on $\mathbb{R}_+$, then by [7, Lemma 2.4], there is a constant $K > 0$ such that

$$G_1(t) \stackrel{\text{def}}{=} F(t)\Big(\int_0^t |g'(s)|\,ds + 1\Big)^{-\omega_2} \le K \quad \text{for} \quad t \in \mathbb{R}_+\,. \tag{3.4}$$

If g is identically constant on $\mathbb{R}_+$, the boundedness of $G_1 = F$ follows from [7, Lemma 2.9]. From this, (1.3), and (3.4), we have

$$\begin{aligned}\int_0^\infty V(t)\,dt &= \int_0^\infty R^{-\beta}F(t)\,dt \\ &= \int_0^\infty R^{-\beta}(t)\Big(\int_0^t |g'(s)|\,ds + 1\Big)^{\omega_2} G_1(t)\,dt \\ &\le K\int_0^\infty R^{-\beta}(t)\Big(\int_0^t |g'(s)|\,ds + 1\Big)^{\omega_2} dt < \infty\end{aligned}$$

by (3.1). Hence, y is of the strong nonlinear limit-circle type and part (i) is proved.

(ii) If (3.3) holds, then part (i) implies equation (1.2) is of the strong nonlinear limit-circle type. Let

$$\int_0^\infty R^{-\beta}(t)\,dt = \infty\,.$$

Then, by [7, Lemma 2.2], there is a solution y of (1.1), a constant $C_0 > 0$, and a $t_0 \in \mathbb{R}_+$ such that

$$0 < \frac{3}{4}C_0 \le F(t) \quad \text{for} \quad t \ge t_0\,. \tag{3.5}$$

From this and (1.3),

$$\begin{aligned}&\gamma\int_{t_0}^\infty |y(t)|^{\lambda+1}\,dt + \int_{t_0}^\infty R^{-1}(t)|y^{[1]}(t)|^{\delta}\,dt \\ &\qquad\qquad = \int_{t_0}^\infty R^{-\beta}(t)F(t)\,dt \ge \frac{3}{4}C_0\int_{t_0}^\infty R^{-\beta}(t)\,dt = \infty\,.\end{aligned}$$

Hence, either $\int\limits_0^\infty |y(t)|^{\lambda+1}\,dt = \infty$ or $\int\limits_0^\infty R^{-1}(t)|y^{[1]}(t)|^{\delta}\,dt = \infty$ and so y is not of the strong nonlinear limit-circle type. This proves (ii). $\square$

Remark 3.2. A result similar to Theorem 3.1(i), but for equation (1.1), is proved in [6, Theorem 1 and Remark 1] if

$$\int_0^\infty R^{-\omega}(t)|e(t)|\,dt < \infty \tag{3.6}$$

holds and ω_2 in (3.1) is replaced by q_1. The following theorem provides a strong nonlinear limit-circle result for equation (1.1) without assuming condition (3.6) holds.

Theorem 3.3. *If*

$$\int_0^\infty R^{-\beta}(t)A^{q_1}(t)\,dt < \infty, \tag{3.7}$$

then equation (1.1) *is of the strong nonlinear limit-circle type.*

Proof. Let y be a solution of (1.1). By Lemma 2.2, the function G is bounded, say $0 \le G(t) \le G_0 < \infty$ for $t \in \mathbb{R}_+$. From this, (1.6), and (3.7), we have

$$\begin{aligned}\int_0^\infty V(t)\,dt = \gamma\int_0^\infty |y(t)|^{\lambda+1}\,dt + \int_0^\infty R^{-1}(t)|y^{[1]}(t)|^{\delta}\,dt\\ = \int_0^\infty R^{-\beta}(t)F(t)\,dt = \int_0^\infty R^{-\beta}(t)A^{q_1}(t)G(t)\,dt < \infty\,,\end{aligned}$$

so y is of the strong nonlinear limit-circle type. □

Remark 3.4. (i) For equation (1.2), note that $q = \omega_1$ if and only if $p \ge \dfrac{\lambda}{\lambda+2}$; in this case $q_1 = \omega_2$. Hence, if $p \ge \dfrac{\lambda}{\lambda+2}$, then Theorem 3.1(i) is a special case of Theorem 3.3. If $p < \dfrac{\lambda}{\lambda+2}$, then $q_1 > \omega_2$ and the conditions in Theorem 3.1 are weaker than those in Theorem 3.3. In this case, (1.6) and (3.4) imply $\lim\limits_{t\to\infty} G(t) = 0$.

(ii) If $p < \dfrac{\lambda}{\lambda+2}$, so that $q_1 > \omega_2$, then condition (3.1) above is better than the condition used in Theorem 1 and Remark 1 in [6] where a result similar to Theorem 3.1(i) above was proved for the forced equation (1.1).

(iii) Consider Equation (1.2) with (3.2) holding. Then $G(t) \ge F(t)\Big(\int\limits_0^\infty |g'(s)|\,ds + 1\Big)^{-q_1}$, and according to [7, Lemma 2.2], there is a solution y, a constant $C > 0$, and $t_0 \in \mathbb{R}_+$ such that (3.5) holds. Hence, $\liminf\limits_{t\to\infty} G(t) > 0$ and so in general the boundedness of G cannot be replaced by $\lim\limits_{t\to\infty} G(t) = 0$ in Lemma 2.2.

4 Non Strong Limit-circle Type Equations

In this section we give conditions that ensure equation (1.1) is not of the strong nonlinear limit-circle type.

Theorem 4.1. *Assume that*

$$\int_0^\infty |e(t)| \exp\Big\{ \frac{p}{p+1} \int_0^t \frac{R'_-(s)}{R(s)}\, ds \Big\}\, dt < \infty\,. \tag{4.1}$$

If

$$\int_0^\infty \exp\Big\{ -\int_0^t \frac{R'_+(s)}{R(s)}\, ds \Big\}\, dt = \infty\,, \tag{4.2}$$

then (1.1) *is not of the strong nonlinear limit-circle type.*

Proof. For any solution y of (1.1), we see that

$$\begin{aligned} \frac{|e(t)|}{r(t)} |y'(t)| = \frac{|e(t)|}{R(t)} |y^{[1]}(t)|^{1/p} &= |e(t)| R^{-1/\delta}(t) \Big(\frac{|y^{[1]}(t)|^\delta}{R(t)} \Big)^{1/(p+1)} \\ &\le |e(t)| R^{-1/\delta}(t) V^{1/(p+1)}(t). \end{aligned} \tag{4.3}$$

A straight forward calculation gives

$$V'(t) = \Big(\frac{1}{R(t)} \Big)' |y^{[1]}(t)|^\delta + \delta \frac{e(t)}{r(t)}\, y'(t)\,.$$

Hence, in view of (4.3),

$$V'(t) \ge -\frac{R'_+(t)}{R(t)}\, V(t) - |e(t)| R^{-1/\delta}(t) V^{1/(p+1)}(t)\,.$$

Setting $Z = V^{1/\delta}$, we obtain

$$Z'(t) + \frac{R'_+(t)}{\delta R(t)}\, Z(t) \ge -\frac{|e(t)|}{\delta}\, R^{-1/\delta}(t),$$

and so

$$\begin{aligned} \Big(Z(t) \exp\Big\{ \frac{1}{\delta} \int_0^t K(s)\, ds \Big\} \Big)' &\ge -\frac{1}{\delta} |e(t)| R^{-1/\delta}(t) \exp\Big\{ \frac{1}{\delta} \int_0^t K(s)\, ds \Big\} \\ &= -\frac{1}{\delta} |e(t)| R^{-1/\delta}(0) \exp\Big\{ \frac{1}{\delta} \int_0^t \frac{R'_-(s)}{R(s)}\, ds \Big\}, \end{aligned}$$

where $K(t) = \dfrac{R'_+(t)}{R(t)}$. Integrating and applying (4.1), we have

$$\begin{aligned} &Z(t)\exp\Big\{\frac{1}{\delta}\int_0^t K(s)\,ds\Big\} \\ &\quad \geq Z(0) - \frac{1}{\delta}R^{-1/\delta}(0)\int_0^\infty |e(t)|\exp\Big\{\frac{1}{\delta}\int_0^t \frac{R'_-(s)}{R(s)}\,ds\Big\}\,dt > -\infty \end{aligned} \tag{4.4}$$

for $t \in \mathbb{R}_+$.

Now, let y be a solution of (1.1) such that

$$Z(0) \geq \frac{1}{\delta}R^{-1/\delta}(0)\int_0^\infty |e(t)|\exp\Big\{\frac{1}{\delta}\int_0^t \frac{R'_-(s)}{R(s)}\,ds\Big\}\,dt + 1\,.$$

From this and (4.4), we obtain

$$V(t) = Z^\delta(t) \geq \exp\Big\{-\int_0^t K(s)\,ds\Big\}, \quad t \in \mathbb{R}_+\,.$$

Integrating and applying condition (4.2), we see that

$$\int_0^t V(s)\,ds = \int_0^t \Big[\frac{|y^{[1]}(s)|^\delta}{R(s)} + \gamma|y(s)|^{\lambda+1}\Big]\,ds \to \infty \tag{4.5}$$

as $t \to \infty$, and hence, y is not of the strong nonlinear limit-circle type. This completes the proof of the theorem. □

In order to interpret the content of the above theorem, we first recall the definitions of nonlinear limit-point and nonlinear limit-circle solutions as well as that of a strong nonlinear limit-point solution.

Definition 4.2. A solution y of (1.1) is said to be of the nonlinear limit-circle type if

$$\int_0^\infty |y(t)|^{\lambda+1}dt < \infty\,, \tag{NLC}$$

and it is said to be of the nonlinear limit-point type otherwise, i.e., if

$$\int_0^\infty |y(t)|^{\lambda+1}dt = \infty\,. \tag{NLP}$$

Equation (1.1) will be said to be of the nonlinear limit-circle type if every solution y of (1.1) satisfies (NLC) and to be of the nonlinear limit-point type if there is at least one solution y for which (NLP) holds.

The definition of a strong nonlinear limit-point solution was first presented in [5].

Definition 4.3. A solution y of (1.1) is said to be of the strong nonlinear limit-point type if

$$\int_0^\infty |y(t)|^{\lambda+1}\,dt = \infty \quad \text{and} \quad \int_0^\infty \frac{|y_2(t)|^\delta}{R(t)}\,dt = \infty\,. \tag{SNLP}$$

Equation (1.1) is said to be of the strong nonlinear limit-point type if every nontrivial solution is of the strong nonlinear limit-point type.

The properties defined above are nonlinear generalizations of the well known limit-point/limit-circle properties introduced by Weyl [15] more than 100 years ago for second order linear equations. Weyl's work has generated a great deal of interest over the last century. The nonlinear limit-point/limit-circle problem originated in the work of Graef [10, 11] and Graef and Spikes [12]. For the history and a survey of what is known about the linear and nonlinear problems as well as their relationships to other properties of solutions such as boundedness, oscillation, and convergence to zero, we refer the reader to the monograph by Bartušek, Došlá, and Graef [1] as well as the recent papers of Bartušek and Graef [2–7] and others.

Now in the proof of Theorem 4.1, we showed that (4.5) holds. Notice that this does not prevent y from being a nonlinear limit-circle type solution as defined in Definition 4.2 above. It is also possible that y is a strong nonlinear limit-point type solution (see Definition 4.3).

Corollary 4.4. (i) *Let*

$$\int_0^\infty \frac{R'_-(t)}{R(t)}\,dt < \infty\,, \quad \int_0^\infty R^{-1}(t)\,dt = \infty, \quad \textit{and} \quad \int_0^\infty |e(t)|\,dt < \infty\,. \tag{4.6}$$

Then (1.1) *is not of the strong nonlinear limit-circle type.*

(ii) *Let*

$$\int_0^\infty \frac{R'_+(t)}{R(t)}\,dt < \infty \quad \textit{and} \quad \int_0^\infty R^{-1/\delta}(t)|e(t)|\,dt < \infty\,. \tag{4.7}$$

Then (1.1) *is not of the strong nonlinear limit-circle type.*

Proof. (i) The first and the third inequalities in (4.6) imply that (4.1) holds. Also,

$$\begin{aligned}\int_0^\infty \exp\left\{-\int_0^t \frac{R'_+(s)}{R(s)}\,ds\right\}dt &= \int_0^\infty \exp\left\{-\int_0^t \frac{R'(s)}{R(s)}\,ds - \int_0^t \frac{R'_-(s)}{R(s)}\,ds\right\}dt \\ &= \int_0^\infty \exp\left\{-\int_0^t \frac{R'_-(s)}{R(s)}\,ds\right\}\frac{R(0)}{R(t)}\,dt \\ &\geq \exp\left\{-\int_0^1 \frac{R'_-(s)}{R(s)}\,ds\right\}R(0)\int_1^\infty \frac{dt}{R(t)} = \infty\,.\end{aligned}$$

Hence, (4.2) holds and the conclusion follows from Theorem 4.1.

(ii) Now (4.2) follows from the first inequality in (4.7). Since

$$\begin{aligned}\int_0^\infty \big|e(t)\big| \exp\Big\{\frac{p}{p+1}\int_0^t \frac{R'_-(s)}{R(s)}\,ds\Big\}\,dt \\ = \int_0^\infty \big|e(t)\big| \exp\Big\{-\frac{p}{p+1}\int_0^t \frac{R'(s)}{R(s)}\,ds + \frac{p}{p+1}\int_0^t \frac{R'_+(s)}{R(s)}\,ds\Big\}\,dt \\ \le \exp\Big\{\frac{p}{p+1}\int_0^\infty \frac{R'_+(s)}{R(s)}\,ds\Big\}\int_0^\infty \big|e(t)\big|\Big(\frac{R(0)}{R(t)}\Big)^{1/\delta}\,dt < \infty\,,\end{aligned}$$

we see that (4.1) holds, and the conclusion again follows from Theorem 4.1. □

Theorem 4.5. *Assume that there is a positive constant K such that*

$$\int_0^\infty r^{\lambda+1}(t)\,dt < \infty \quad \textit{and} \quad \frac{r(t)}{a(t)} \le K \tag{4.8}$$

for $t \in \mathbb{R}_+$. In addition, assume that one of the following conditions holds:

(i)

$$\int_0^\infty a^{-1/p}(t)\,dt = \infty \quad \textit{and} \quad \int_0^\infty e(t)\,dt = \pm\infty\,;$$

or

(ii) *for $t \in \mathbb{R}_+$,*

$$R(t) \le R_0 < \infty\,, \quad r(t) \le r_0 < \infty\,, \quad |e(t)| \le M < \infty, \tag{4.9}$$

and

$$-\infty \le \liminf_{t\to\infty}\int_0^t e(s)\,ds < \limsup_{t\to\infty}\int_0^t e(s)\,ds \le \infty\,. \tag{4.10}$$

Then no solution of (1.1) *is of the strong nonlinear limit-circle type, and so equation* (1.1) *is not of the strong nonlinear limit-circle type.*

Proof. Let y be a solution of (1.1) and suppose, to the contrary, that it is of the strong nonlinear limit-circle type. Then Definition 1.2 and the second inequality in (4.8) imply

$$\int_0^\infty |y(t)|^{\lambda+1}\,dt < \infty \quad \text{and} \quad \int_0^\infty |y'(t)|^{p+1}\,dt < \infty\,. \tag{4.11}$$

From this and Sz.-Nagy's inequality (see [9, Chap. V, Theorem 1]), y is bounded on $\mathbb{R}_+$, say

$$|y(t)| \le M_1\,, \quad t \in \mathbb{R}_+\,. \tag{4.12}$$

Hence, (4.8), (4.11), and Hölder's inequality imply

$$\int_0^\infty r(t)|y(t)|^\lambda\,dt \le \Big(\int_0^\infty |y(t)|^{\lambda+1}\,dt\Big)^{\frac{\lambda}{\lambda+1}}\Big(\int_0^\infty r^{\lambda+1}(t)\,dt\Big)^{\frac{1}{\lambda+1}} < \infty,$$

and we see that

$$\int_0^\infty r(t)|y(t)|^\lambda \operatorname{sgn} y(t)\,dt = M_2 \in \mathbb{R}\,. \tag{4.13}$$

An integration of (1.1) on $[0, t]$ gives

$$y^{[1]}(t) = y^{[1]}(0) - \int_0^t r(s)|y(s)|^\lambda \operatorname{sgn} y(s)\,ds + \int_0^t e(s)\,ds\,. \tag{4.14}$$

(i) Suppose $\int\limits_0^\infty e(t)\,dt = \infty$; the case $\int\limits_0^\infty e(t)\,dt = -\infty$ can be handled similarly. Then (4.13)–(4.14) imply

$$\lim_{t\to\infty} y^{[1]}(t) = \infty\,.$$

Hence, from this and the hypotheses in part (i),

$$y(t) - y(0) = \int_0^t y'(s)\,ds = \int_0^t a^{-1/p}(s)|y^{[1]}(s)|^{1/p} \operatorname{sgn} y^{[1]}(s)\,ds \to \infty \quad \text{as} \quad t \to \infty\,.$$

This contradicts (4.12) and proves part (i).

(ii) Since $R(t) \le R_0$, Definition 1.2 implies

$$\int_0^\infty |y^{[1]}(t)|^\delta\,dt < \infty\,. \tag{4.15}$$

From (1.1), (4.9), and (4.12), we see that

$$\big(y^{[1]}(t)\big)' \quad \text{is bounded on} \quad \mathbb{R}_+\,.$$

Applying Lemma 2.3 to (4.15) with $f = y^{[1]}$ and $m = \delta$ gives

$$\lim_{t\to\infty} y^{[1]}(t) = 0\,.$$

Then (4.13) and (4.14) imply $\lim\limits_{t\to\infty} \int_{t_0}^t e(s)\,ds$ exists, which contradicts (4.10). This proves part (ii) and completes the proof of the theorem. □

The results in this section can be reformulated as giving necessary conditions for equation (1.1) to have a strong nonlinear limit-circle solution. For example, Theorem 4.1 could be presented as follows.

Theorem 4.6. *Assume* (4.1) *holds. If equation* (1.1) *has a strong nonlinear limit-circle solution, then*

$$\int_0^\infty \exp\Big\{ - \int_0^t \frac{R'_+(s)}{R(s)}\,ds \Big\}\,dt < \infty\,.$$

5 Applications and Examples

The following application of Lemma 2.2 gives global estimates on solutions of equation (1.1) and their derivatives.

Theorem 5.1. *Let y be a solution of* (1.1). *Then there are positive constants C and C_1 such that*

$$|y(t)| \le C R^{-\omega}(t) A^{q_1/(\lambda+1)}(t)$$

and

$$|y'(t)| \le C_1 a^{-\frac{1}{p}}(t) R^{\alpha/(p+1)}(t) A^{q_1/(p+1)}(t)$$

on $\mathbb{R}_+$.

The following examples show that Theorems 4.1 and 4.5 are independent of each other even in case $R(t) \le R_0 < \infty$.

Example 5.2. Consider a special case of (1.1), namely,

$$y'' + t^{-7}(2+\sin t) y^3 = \sin t\,, \quad t \ge 1. \tag{5.1}$$

Here, $p = 1$, $\lambda = 3$, and $a \equiv 1$. The hypotheses of Theorem 4.5(ii) are satisfied, so equation (5.1) is not of the strong nonlinear limit-circle type. Theorem 4.1 cannot be applied since (4.2) does not hold. To see this, let $k_0 \in \{1, 2, \dots\}$ and let $\{c_k\}_{k=k_0}^{\infty}$, and $\{d_k\}_{k=k_0}^{\infty}$ be sequences such that $c_k \in \big(\frac{3\pi}{2} + 2k\pi, \frac{5}{3}\pi + 2k\pi\big)$, $d_k \in \big(\frac{7}{3}\pi + 2k\pi, \frac{5\pi}{2} + 2k\pi\big)$, $\frac{\cos t}{3} - \frac{7}{c_{k_0}} \ge 0$ on $[c_k, d_k]$, $k = k_0, k_0+1, \dots$, and let $K = \frac{1}{3} - \frac{7\pi}{c_{k_0}} > 0$. Then,

$$\int_{c_k}^{d_k} \frac{R'_+(t)}{R(t)}\,dt \ge \int_{c_k}^{d_k} \Big(\frac{\cos t}{3} - \frac{7}{c_{k_0}}\Big)\,dt \ge \frac{1}{3} - \frac{7}{c_{k_0}}(d_k - c_k) \ge \frac{1}{3} - \frac{7\pi}{c_{k_0}} = K$$

and

$$\begin{aligned} \int_0^\infty \exp\Big\{-\int_0^t \frac{R'_+(s)}{R(s)}\,ds\Big\}\,dt &\le C + \sum_{k=k_0}^{\infty} \exp\Big\{-\sum_{i=k_0}^{k} \int_{c_i}^{d_i} \frac{R'_+(s)}{R(s)}\,ds\Big\} \\ &\le C + \sum_{k=k_0}^{\infty} \exp\Big\{-K(k-k_0)\Big\} < \infty, \end{aligned}$$

where $C = \int_0^{c_{k_0}} \exp\big\{-\int_0^t \frac{R'_+(s)}{R(s)}\,ds\big\}\,dt$. Thus, (4.2) does not hold.

The following example shows that strong nonlinear limit-circle solutions may exist even if R is positive and small and $\int_0^\infty e(t)\,dt$ exists.

Example 5.3. Consider the equation

$$y'' + t^{-1/4}y^3 = -t^{-3/4}\sin t - \frac{3}{2}t^{-7/4}\cos t + \frac{21}{16}t^{-11/4}\sin t + t^{-5/2}\sin^3 t\,.$$

We have $p = 1$, $\lambda = 3$, $a \equiv 1$, and $r(t) = t^{-1/4}$. This equation has the strong nonlinear limit-circle solution $y(t) = t^{-3/4}\sin t$, $t \geq 1$. Note that the condition $\int_0^\infty r^{\lambda+1}(t)\,dt < \infty$ in Theorem 4.5 is not satisfied. In addition, we see that $\int_0^\infty e(t)\,dt$ is convergent by [13, Paragraph 3.761].

We conclude this paper with one final example.

Example 5.4. Consider the equation

$$\big(|y'|^{p-1}y'\big)' + t^v|y|^\lambda \operatorname{sgn} y = 0. \tag{5.2}$$

If $v \geq -\frac{1}{\alpha}$, then (3.2) holds, and so by Theorem 3.1(ii), equation (5.2) is of the strong nonlinear limit circle type if and only if $v > \frac{(\lambda+2)p+1}{(\lambda+1)p} = \frac{1}{\beta}$. If $v < -\frac{1}{\alpha}$, then Theorem 4.1 implies (5.2) is not of the nonlinear limit-circle type. Hence, in summary, (5.2) is of the strong nonlinear limit-circle type if and only if $v > \frac{1}{\beta}$.

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References

[1] M. Bartušek, Z. Došlá, and J. R. Graef, *The Nonlinear Limit-Point/Limit-Circle Problem*, Birkhäuser, Boston, 2004.

[2] M. Bartušek and J. R. Graef, Asymptotic behavior of solutions of a differential equation with p-Laplacian and a forcing term, *Differ. Equ. Dyn. Syst.* **15** (2007), 61–87.

[3] M. Bartušek and J. R. Graef, The strong nonlinear limit-point/limit-circle properties for super-half-linear equations, *Panamer. Math. J.* **17** (2007), 25–38.

[4] M. Bartušek and J. R. Graef, The strong nonlinear limit-point/limit-circle properties for sub-half-linear equations, *Dynam. Systems Appl.* **15** (2006), 585–602.

[5] M. Bartušek and J. R. Graef, The strong nonlinear limit-point property for Emden-Fowler equations, *Differ. Equ. Dyn. Syst.* **14** (2006), 383–405.

[6] M. Bartušek and J. R. Graef, Asymptotic properties of solutions of a forced second order differential equation with p-Laplacian, Panamer. Math. J. 16 (2006), 41–59.

[7] M. Bartušek and J. R. Graef, The nonlinear limit-point/limit-circle problem for second order equations with p-Laplacian, Dynam. Systems Appl. 14 (2005), 431–446.

[8] M. Bartušek and E. Pekárková, On existence of proper solutions of quasilinear second order differential equations, Electron. J. Qual. Theory Differ. Equ. 2007 (2007), No. 5, 1-14.

[9] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, 1961.

[10] J. R. Graef, Limit circle criteria and related properties for nonlinear equations, *J. Differential Equations* **35** (1980), 319–338.

[11] J. R. Graef, Limit circle type results for sublinear equations, *Pacific J. Math.* **104** (1983), 85–94.

[12] J. R. Graef and P. W. Spikes, On the nonlinear limit–point/limit–circle problem, *Nonlinear Anal.* **7** (1983), 851–871.

[13] I. S. Grandshteyn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Acad. Press., Boston, 1994.

[14] J. D. Mirzov, Asymptotic properties of solutions of systems of nonlinear nonautonomous ordinary differential equations, Folia FSN Univ. Mas. Brun., Brno, Math. 14, 2004.

[15] H. Weyl, Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen, *Math. Ann.* **68** (1910), 220–269.