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Some Hardy Type Inequalities with Weighted Functions via Opial Type Inequalities

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Abstract

In this paper, we will prove several new inequalities of Hardy type with explicit constants. The main results will be proved using generalizations of Opial's inequality.

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1 Introduction

The classical Hardy inequality (see [10]) states that for $f \geq 0$ integrable over any finite interval $(0, x)$ and f^p integrable and convergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

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The constant $(p/(p-1))^p$ is the best possible. Some extensions of Hardy's inequality were considered in Beesack [5].

Our aim in this paper is to prove some inequalities with weighted functions of Hardy type using Opial type inequalities.

2 Main Results

Throughout the paper, all functions are assumed to be positive and measurable and all the integrals which appear in the inequalities are assumed to exist and be finite.

To obtain inequalities of Hardy type we look at inequalities for

$$\int_a^b R(x,b)F(x)F'(x)dx,$$

where $R(x,b) = \int_x^b r(t)\,dt$ and $F(x) = \int_a^x f(t)dt$. Each Opial type inequality will give a Hardy type inequality . We will use a number of Opial type inequalities to illustrate this point.

Boyd and Wong [8] proved if $p > 0$ and if y is an absolutely continuous function on $[a,b]$ with $y(a) = 0$ (or $y(b) = 0$), then

$$\int_a^b q(t)\,|y(t)|^p\,|y'(t)|\,dt \leq \frac{1}{\lambda_0(p+1)}\int_a^b w(t)\,|y'(t)|^{p+1}\,dt, \tag{2.1}$$

where q and w are nonnegative functions in $C^1[a,b]$, and such that the boundary value problem has a solution

$$(q(t)\,(u'(t))^p)' = \lambda w'(t)u^p(t),$$

with $u(a) = 0$ and $q(b)\,[u'(b)]^p = \lambda\, w(b)\, u^p(b)$, for which $u' > 0$ in $[a,b]$ (let λ_0 be the smallest eigenvalue of the boundary value problem).

Applying the inequality (2.1) on the term $(p+1)\int_a^b R(x,b)F^p(x)F'(x)dx$, we have

$$(p+1)\int_a^b R(x,b)F^p(x)F'(x)dx \leq \frac{1}{\lambda_0}\int_a^b s(t)(F'(x))^{p+1}dx, \tag{2.2}$$

where r and s are nonnegative functions, $r \in C[a,b]$, $s \in C^1[a,b]$, and such that the boundary value problem has a solution

$$(R(x,b)\,(u'(x))^p)' = \lambda s'(x)u^p(x), \tag{2.3}$$

($R(x,b) = \int_x^b r(t)dt$ and note $R \in C^1[a,b]$ since $r \in C[a,b]$) with $u(a) = 0$ and $u(b) = 0$, for which $u' > 0$ in $[a,b]$ (let λ_0 be the smallest eigenvalue of the boundary value problem).

Theorem 2.1. *Assume that r, s are nonnegative functions with $r \in C[a,b]$, $s \in C^1[a,b]$ and $p > 0$. Then*

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx \leq \frac{1}{\lambda_0}\int_a^b s(x)\,(f(x))^{p+1}\,dx,$$

for all integrable functions $f \geq 0$ where λ_0 is the smallest eigenvalue of the boundary value problem (2.3).

Proof. Let $F(x) = \displaystyle\int_a^x f(t)dt$. Since f is integrable on $[a,b]$ then F is absolutely continuous on $[a,b]$. Note $F(a) = 0$, $F'(x) = f(x) > 0$ and

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx = \int_a^b r(x)F^{p+1}(x)dx.$$

Integration by parts gives

$$\begin{aligned}\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx &= -R(x,b)F^{p+1}(x)\big|_a^b \\ &\quad + (p+1)\int_a^b R(x,b)F^p(x)F'(x)dx\end{aligned}$$

where $R(x,b) = \displaystyle\int_x^b r(t)dt$. Using $R(b,b) = 0$ and $F(a) = 0$, we have

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx = (p+1)\int_a^b R(x,b)F^p(x)F'(x)dx. \tag{2.4}$$

Now (2.2) establishes the result. □

Boyd in [7] extended the results of [8]. In [7, Theorem 2.1] the author established inequalities (best possible constants) of the form

$$\int_a^b s(t)\,|y(t)|^p\,|y'(t)|^q\,dt \leq \frac{k}{\lambda_0(p+q)}\left(\int_a^b r(t)\,|y'(t)|^k\,dt\right)^{\frac{p+q}{k}},$$

where $p > 0$, $k > 1$, $0 \leq q \leq k$ with $r, s \in C^1(a,b)$ and $r > 0$, $s > 0$ a.e. on (a,b); here λ_0 is the smallest eigenvalue of an appropriate boundary value problem (assuming certain conditions are satisfied; see [7]). With these conditions (with $q = 1$ and $k > 1$) we obtain using the procedure before and in Theorem 2.1

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx \leq \frac{k}{\lambda_0}\left(\int_a^b r(t)\,(f(t))^k dt\right)^{\frac{p+1}{k}},$$

where λ_0 is the smallest eigenvalue of an appropriate boundary value problem.

Instead of this inequality (and presenting the conditions to guarantee the existence of λ_0) we will consider two special cases of this result, one found in [7] and the other in [6].

In the following, we apply an inequality due to Boyd [7] and the Hölder inequality. The Boyd inequality states that: If y is absolutely continuous on $[a,b]$ with $y(a)=0$ (or $y(b)=0$), then

$$\int_a^b |y(t)|^\nu \, |y'(t)|^\eta \, dt \le N(\nu,\eta,s)(b-a)^\nu \left(\int_a^b |y'(t)|^s \, dt\right)^{\frac{\nu+\eta}{s}}, \tag{2.5}$$

where $\nu > 0$, $s > 1$, $0 \le \eta < s$,

$$N(\nu,\eta,s) := \frac{(s-\eta)\,\nu^\nu \sigma^{\nu+\eta-s}}{(s-1)(\nu+\eta)\left(I(\nu,\eta,s)\right)^\nu}, \; \sigma := \left\{\frac{\nu(s-1)+(s-\eta)}{(s-1)(\nu+\eta)}\right\}^{\frac{1}{s}}, \tag{2.6}$$

and

$$I(\nu,\eta,s) := \int_0^1 \left\{1+\frac{s(\eta-1)}{s-\eta}t\right\}^{-(\nu+\eta+s\nu)/s\nu} [1+(\eta-1)t]t^{1/\nu-1}dt.$$

Apply the Hölder inequality and inequality (2.5) to obtain

$$\begin{aligned}
\int_a^b R(x,b)F^p(x)F'(x)dx &\le \left(\int_a^b R^p(x,b)dx\right)^{\frac{1}{p}} \left(\int_a^b F^{pq}(x)\left(F'(x)\right)^q dx\right)^{\frac{1}{q}} \\
&\le N^{\frac{1}{q}}(pq,q,s)(b-a)^p \left(\int_a^b R^p(x,b)dx\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_a^b \left(F'(x)\right)^s dx\right)^{\frac{p+1}{s}},
\end{aligned} \tag{2.7}$$

where $p > 1$, $1/p+1/q = 1$, $s > 1$ and $1 < q < s$; here $N(pq,q,s)$ is determined from (2.6) by putting $\nu = pq$ and $\eta = q$.

Theorem 2.2. *Assume that r is a nonnegative measurable function on (a,b), $p > 1$, $s > 1$, $1 < q < s$ and $1/p+1/q = 1$. Then*

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx \le C\left(\int_a^b (f(x))^s \, dx\right)^{\frac{p+1}{s}},$$

for all integrable functions $f \ge 0$; here

$$C = (p+1)\, N^{\frac{1}{q}}(pq,q,s)(b-a)^p \left(\int_a^b R^p(x,b)dx\right)^{\frac{1}{p}}.$$

Proof. The result follows from (2.4) and (2.7). $\square$

As in the proof of Theorem 2.1, by putting $F(x) = \int_x^b f(t)dt$, we have the following result.

Theorem 2.3. *Assume that r is a nonnegative measurable function on (a, b), $p > 1$, $1 < q < s$ and $1/p + 1/q = 1$. Then*

$$\int_a^b r(x) \left(\int_x^b f(t)dt \right)^{p+1} dx \leq C \left(\int_a^b (f(x))^s \, dx \right)^{\frac{p+1}{s}},$$

for all integrable functions $f \geq 0$; here

$$C = (p+1) \, N^{\frac{1}{q}}(pq, q, s)(b-a)^p \left(\int_a^b R^p(a, x)dx \right)^{\frac{1}{p}} \text{ and } R(a, x) = \int_a^x r(t)dt.$$

When $\eta = s$ equation (2.5) becomes

$$\int_a^b |y(t)|^\nu \, |y'(t)|^\eta \, dt \leq L(\nu, \eta)(b-a)^\nu \left(\int_a^b |y'(t)|^\eta \, dt \right)^{\frac{\nu+\eta}{\eta}}, \tag{2.8}$$

where

$$L(\nu, \eta) := \frac{\eta \nu^\eta}{\nu + \eta} \left(\frac{\nu}{\nu + \eta} \right)^{\frac{\nu}{\eta}} \left(\frac{\Gamma\left(\frac{\eta+1}{\eta} + \frac{1}{\nu}\right)}{\Gamma\left(\frac{\eta+1}{\eta}\right) \Gamma\left(\frac{1}{\nu}\right)} \right)^\nu, \tag{2.9}$$

and Γ is the Gamma function. Apply inequality (2.8) to obtain

$$\int_a^b F^{pq}(x) \left(F'(x)\right)^q dx \leq L(pq, q)(b-a)^{pq} \left(\int_a^b \left(F'(x)\right)^q dx \right)^{\frac{pq+q}{q}}, \tag{2.10}$$

where

$$L(pq, q) = \frac{(pq)^q}{p+1} \left(\frac{p}{p+1} \right)^p \left(\frac{\Gamma\left(\frac{q+1}{q} + \frac{1}{pq}\right)}{\Gamma\left(\frac{q+1}{q}\right) \Gamma\left(\frac{1}{pq}\right)} \right). \tag{2.11}$$

Using (2.10), we see that

$$\begin{aligned} \int_a^b R(x, b) F^p(x) F'(x) dx \; &\leq \; \left(\int_a^b R^p(x, b)dx \right)^{\frac{1}{p}} \left(\int_a^b F^{pq}(x) \left(F'(x)\right)^q dx \right)^{\frac{1}{q}} \\ &\leq \; L^{\frac{1}{q}}(pq, q)(b-a)^p \left(\int_a^b R^p(x, b)dx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_a^b \left(F'(x)\right)^q dx \right)^{\frac{p+1}{q}}, \end{aligned}$$

where $p > 1$ and $1/p + 1/q = 1$. This gives us the following results.

Theorem 2.4. *Assume that r is a nonnegative measurable function on (a,b), $p>1$, $q>1$ and $1/p+1/q=1$. Then*

$$\int_a^b r(x)\left(\int_a^x f(t)dt\right)^{p+1} dx \le C\left(\int_a^b (f(x))^q\,dx\right)^{\frac{p+1}{q}},$$

for all integrable functions $f\ge 0$; here

$$C=(p+1)\,L^{\frac{1}{q}}(pq,q)(b-a)^p\left(\int_a^b R^p(x,b)dx\right)^{\frac{1}{p}}$$

and $L(pq,q)$ is defined as in (2.11).

Theorem 2.5. *Assume that r is a nonnegative measurable function on (a,b), $p>1$, $q>1$ and $1/p+1/q=1$. Then*

$$\int_a^b r(x)\left(\int_x^b f(t)dt\right)^{p+1} dx \le C\left(\int_a^b (f(x))^q\,dx\right)^{\frac{p+1}{q}},$$

for all integrable functions $f\ge 0$; here

$$C=(p+1)\,L^{\frac{1}{q}}(pq,q)(b-a)^p\left(\int_a^b R^p(a,x)dx\right)^{\frac{1}{p}}$$

and $L(pq,q)$ is defined as in (2.11).

Finally we apply an Opial type inequality due to Beesack [6] to prove inequalities of Hardy type. The inequality due to Beesack is given in the following theorem.

Theorem 2.6. *Let r,s be nonnegative, measurable functions on (α,τ). Further assume that $k>1$, $p>0$, $0<q<k$, and let y be absolutely continuous in $[\alpha,\tau]$ such that $y(\alpha)=0$. Then*

$$\int_\alpha^\tau r(t)\left|y(t)\right|^p\left|y'(t)\right|^q dt \le K_1(p,q,k)\left[\int_\alpha^\tau s(t)\left|y'(t)\right|^k dt\right]^{(p+q)/k}, \tag{2.12}$$

where

$$\begin{aligned} K_1(p,q,k) &= \left(\frac{q}{q+p}\right)^{\frac{q}{k}} \\ &\times\left(\int_\alpha^\tau (r(y))^{\frac{k}{k-q}}(s(y))^{-\frac{q}{k-q}}\left(\int_a^y s^{\frac{-1}{k-1}}(t)dt\right)^{p(k-1)/(k-q)} dy\right)^{\frac{k-q}{k}}. \end{aligned}$$

If instead $[\alpha, \tau]$ *is replaced by* $[\tau, \beta]$ *and* $y(\alpha) = 0$ *is replaced by* $y(\beta) = 0$*, then*

$$\int_{\tau}^{\beta} r(t) \, |y(t)|^p \, |y'(t)|^q \, dt \leq K_2(p, q, k) \left[\int_{\tau}^{\beta} s(t) \, |y'(t)|^k \, dt \right]^{(p+q)/k}, \tag{2.13}$$

where

$$\begin{aligned} K_2(p, q, k) &= \left(\frac{q}{q+p} \right)^{\frac{q}{k}} \\ &\quad \times \left(\int_{\tau}^{\beta} (r(y))^{\frac{k}{k-q}} (s(y))^{-\frac{q}{k-q}} \left(\int_{y}^{\beta} s^{\frac{-1}{k-1}}(t) dt \right)^{p(k-1)/(k-q)} dy \right)^{\frac{k-q}{k}}. \end{aligned}$$

Now, we apply inequality (2.12) and (2.13). For completeness we apply (2.12) with $k > 1$ to obtain

$$\int_{a}^{b} R(x, b) \, F^p(x) \, F'(x) dx \leq K_1(p, 1, k) \left[\int_{a}^{b} s(x) (F'(x))^k dx \right]^{(p+1)/k}, \tag{2.14}$$

where

$$\begin{aligned} K_1(p, 1, k) = \left(\frac{1}{1+p} \right)^{\frac{1}{k}} \\ \times \left(\int_{a}^{b} (R(x, b))^{\frac{k}{k-1}} (s(x))^{-\frac{1}{k-1}} \left(\int_{a}^{x} s^{\frac{-1}{k-1}}(t) dt \right)^{p} dx \right)^{\frac{k-1}{k}}. \end{aligned} \tag{2.15}$$

Theorem 2.7. *Let* $p > 0$, $k > 1$ *and let* r, s *be nonnegative measurable functions on* (a, b)*. Then*

$$\int_{a}^{b} r(x) \left(\int_{a}^{x} f(t) dt \right)^{p+1} dx \leq (p+1) \, K_1(p, 1, k) \left[\int_{a}^{b} s(x) (f(x))^k dx \right]^{(p+1)/k},$$

for all integrable functions $f \geq 0$*; here* $K_1(p, 1, k)$ *is defined as in* (2.15).

Proof. The result follows from (2.4) and (2.14). $\square$

The proof of the following theorem can be obtained by applying inequality (2.13) and hence is omitted.

Theorem 2.8. *Let* $p > 0$, $k > 1$ *and let* r, s *be nonnegative measurable functions on* (a, b)*. Then*

$$\int_{a}^{b} r(x) \left(\int_{x}^{b} f(t) dt \right)^{p+1} dx \leq (p+1) \, K_2(p, 1, k) \left[\int_{a}^{b} s(x) (f(x))^k dx \right]^{(p+1)/k},$$

for all integrable functions $f \geq 0$*; here*

$$
\begin{aligned}
K_2(p,1,k) &= \left(\frac{1}{1+p}\right)^{\frac{1}{k}} \\
&\quad \times \left(\int_a^b (R(a,x))^{\frac{k}{k-1}} (s(x))^{-\frac{1}{k-1}} \left(\int_x^b s^{\frac{-1}{k-1}}(t)dt\right)^p dx\right)^{\frac{k-1}{k}}.
\end{aligned}
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